Algebraic Geometry

Dr Birkar (C.Birkar@dpmms.cam.ac.uk)
Typeset by Aaron Chan(akyc2@cam.ac.uk)

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What is it? Solving systems of equations...(true but misleading)

Let k be algebraic closed field (e.g. $\mathbb{C}$)

Definition 0.1 (Affine space)
The n-dimensional affine space is the geometric object $\mathbb{A}_k^n = \{(x_1, \dots, x_n) \mid x_i \in k\}$.
Corresponding to this we have an algebraic object $k[t_1, \dots, t_n]$, the polynomial ring.
For any ideal $I \trianglelefteq k[t_1, \dots, t_n]$ we define an object, affine algebraic set

$$V(I) = \{p \in \mathbb{A}_k^n \mid f(p) = 0 \quad \forall f \in I\} \tag{0.1}$$

If $I =< f_1, \dots, f_l >$, then $V(I) =$ solution of the system $f_i = 0 i = 1, 2, \dots, l$
Given an affine algebraic set $X \subseteq \mathbb{A}_k^n$, we can define the ideal of X as

$$I_x = \{f \in k[t_1, \dots, t_n] \mid f(p) = 0 \quad \forall p \in X\} \tag{0.2}$$

Theorem 0.2
Suppose $I, J \trianglelefteq \mathrm{k}[\mathrm{t}_1, \dots, \mathrm{t}_\mathrm{n}]$ are ideals and $X, Y \subseteq \mathbb{A}_k^n$ are affine algebraic sets, Then:

(1) $V(I) \cup V(J) = V(IJ) = V(I \cap J)$
(2) $V(I) \cap V(J) = V(I + J)$
(3) $I_{X \cup Y} = I_X \cap I_Y$
(4) $I_{V(I)} = \sqrt{I} = \{f \mid f^\gamma \in I \text{ some } \gamma \in \mathbb{N}\}$ (Remark: $(I \subseteq \sqrt{I})$)

$$\{\text{Radical ideals of } \mathrm{k}[\mathrm{t}_1, \dots, \mathrm{t}_\mathrm{n}]\} \xleftrightarrow{1:1} \{\text{affine algebraic set } \subseteq \mathbb{A}_k^n\} \tag{0.3}$$

For any affine algebraic set $X = V(I)$, we define the coordinate ring of X to be $k[X] = \mathrm{k}[\mathrm{t}_1, \dots, \mathrm{t}_\mathrm{n}] / \sqrt{I}$ (k-algebra with no nilpotent elements)

In general, in a mathematical theory, you need to define the basic object, and then you need to define relations, and you need to classify your objects according to the relations.

Example:
(Group Theory) object: groups; relation: Homomorphism
(Topology) object: Topological spaces; relation: continuous maps
(Algebraic Geometry) object: algebraic set (affine/projective); relation: regular map/morphism/rational maps

Definition 0.3 (Regular map=Morphism)
Let $X \subseteq \mathbb{A}_k^n, Y \subseteq \mathbb{A}_k^m$ be affine algebraic sets. A regular map $\phi : X \to Y$ is given by $\phi = (f_1, \dots, f_m)$ where $f_i \in \mathrm{k[t_1, \dots, t_n]}$ (f_i can be considered as functions on X)
Any such ϕ gives a k-algebra homomorphism:

$$\begin{aligned} \phi^* : k[Y] &\to k[X] && (0.4) \\ f &\mapsto f\phi && (0.5) \end{aligned}$$

Definition 0.4 (Projective space)
The n-dimensional projective space is defined as $\mathbb{P}_k^n = \{(x_0 : \dots : x_n) \mid x_i \in k, \text{ not all zeroes}\}$ where $(x_0 : \dots : x_n)$ represents the class of $(x_0, \dots, x_n)$ where we take $(x_0 : \dots : x_n) = (y_0 : \dots : y_n)$ if $\exists \lambda \in k, \lambda \neq 0$ s.t. $(x_0, \dots, x_n) = \lambda(y_0, \dots, y_n)$

We can include $\mathbb{A}_k^n$ in $\mathbb{P}_k^n$ as follows:
$\mathbb{P}_k^n \supseteq U_i = \{(x_0 : \dots : x_n) \mid x_i \neq 0\} \overset{1:1}{\mapsto} (x_0/x_i, \dots, x_{i-1}/x_i, x_{i+1}/x_i, \dots, x_n/x_i) \in \mathbb{A}_k^n$

Corresponding to $\mathbb{P}_k^n$ we have algebraic object $k[t_0, \dots, t_n] \bigoplus_{d \geq 0} S_d$ which is a graded ring organised by degree of polynomial. (S_d are homogeneous polynomial of degree d)
A homogeneous ideal in $k[t_0, \dots, t_n]$ is an ideal s.t. $I = \bigoplus_{d \geq 0}(I \cap S_d)$ (i.e. I generated by homogeneous polynomials)

Definition 0.5 (Projective algebraic set)
A projective algebraic set in $\mathbb{P}_k^n$ is a set of the form

$$V(I) = \{p \in \mathbb{P}_k^n \mid F(p) = 0 \quad \forall \text{homogeneous } F \in I\} \tag{0.6}$$

Theorem 0.6
$I, J \trianglelefteq k[t_0, \dots, t_n]$ homogeneous ideals. Then,

(1) $V(I) \cup V(J) = V(IJ) = V(I \cap J)$
(2) $V(I) \cap V(J) = V(I + J)$
(3) If $V(I) \neq 0$ then $I_{V(I)} = \sqrt{I}$ (defined similarly as affine case)

Zariski Topology:
The projective (or affine) algebraic set in $\mathbb{P}_k^n$ ($\mathbb{A}_k^n$) form a topology algebraic set considered as closed set

Fact: Any projective algebraic set X can be written as a union $X = X_1 \cup \dots \cup X_l$ where X_i cannot written as $Y \cup Z$ where Y and Z are proper algebraic sets. The X_i are called the irreducible components of X. Such X_i are called (projective) varieties

Definition 0.7 (Rational functions)
Let $X \subseteq \mathbb{P}_k^n$ be a projective varieties. A rational function on X is of the form F/G where F and G are homogeneous polynomial of the same degree s.t. G does not vanish on X identically.

Example
$X = \mathbb{P}_k^2 \qquad f = \frac{t_0 - t_1}{t_2}$, or $f = \frac{at_1^2 - bt_0t_2}{t_0^2}$ (first one not defined on (1:2:0))

Definition 0.8 (Function field)
The set of all rational functions on a projective variety form a field denote $k(X)(\supseteq k)$ and called the function field of X

Definition 0.9 (Dimension)
The dimension of a projective variety X is the transcendence degree of $k(X)$ over k, denote $\dim X$.

$$\dim X = \max l \text{ s.t. we have a sequence of varieties } X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_l \subsetneq X \tag{0.7}$$

If $\dim X = 1$, X is called a curve (see notes for pictures)

Definition 0.10 (Local Rings)
Suppose that X is a projective variety, and $x \in X$, the local ring $O_{X,x}$ is the set of rational functions on X which are defined at x. The ring $\mathcal{O}_{X,x}$ is a local ring, i.e. it has one unique maximal ideal.

Definition 0.11 (Singular point)
A point $x \in X$ is non-singular if $\mathcal{O}_{X,x}$ is a regular ring.

Being non-singular means that there is no dramatic change near x (see notes for pictures)

Definition 0.12 (Divisor)
X projecitve variety, divisor D on X is of the form $D = \sum m_i D_i$ where $m_i \in \mathbb{Z}$ and $D_i \subsetneq X$ are varieties which $\dim D_i = \dim X$ and the summation is finite.

Definition 0.13 (Divisor of rational function)
X projecitve variety, $f \in k(X)$ a rational function. We associate a divisor (f) to f which is defined by (f) =zeroes-poles (zeroes and poles with multiplicities). We also define the divisor class group:

$$\mathrm{Cl}(X) = \frac{\text{group of divisors}}{\text{divisors of rational functions}} \tag{0.8}$$

(i.e. $D = D' + (f) \Rightarrow D = D'$ in $\mathrm{Cl}(X)$)
Given $D = \sum m_i D_i$ define degree of a divisor D by $\deg D = \sum m_i$

Important case: X is non-singular of dim=1. In this case, we have $\deg(f) = 0$ and so $D = D' + (f) \Rightarrow \deg D = \deg D'$

Definition 0.14 (Space of a divisor)
D a divisor on a non-singular projective variety X, we define

$$H^0(X, D) = \{f \in k(X) \mid D + (f) \geq 0\} \cup \{0\}$$

(This is a vector space over k)

Theorem 0.15 (Riemann-Roch Theorem for Curves)
X is a non-singular projective curve, and D ad divisor on X. Then, there is a special divisor on X denoted K_X s.t. for any divisor D on X we have

$$\dim_k H^0(X, D) - \dim_k H^0(X, K_X - D) = \deg D + 1 - g(X) \tag{0.9}$$

where $g(X)$ is a fixed number called the genus of X, defined by

$$g(X) = \dim_k H^0(X, K_X) \tag{0.10}$$

Example:
The varieties in $\mathbb{P}_k^2$ are of the following form: $\emptyset$, point, curves, $\mathbb{P}_k^2$

Any non-singular curve $X \subseteq \mathbb{P}_k^2$ is given by a homogeneous polynomial, i.e. $\exists$ homogeneous F s.t. $X = V(F)$, F irreducible.

A basic fact is that

$$g(X) = \frac{1}{2}(d-1)(d-2) \qquad \text{where } d = \deg F \tag{0.11}$$

In particular, if $d = 1, 2 \Rightarrow g(X) = 0 \Rightarrow X \simeq \mathbb{P}_k^1$

If $d = \deg F = 3$ then $g(X) = 1$ and X is called an elliptic curve.

If $k = \mathbb{C}$, then X is a Riemann surface and $g(X)$ is the same as numbers of holes in X.

$\deg F = 3$, i.e. X and elliptic curve, from Riemann-Roch (0.15), we can see that $\deg K_X = 2g(X) - 2$, so in this case $\deg K_X = 0 [\Rightarrow K_X = 0$ in $\mathrm{Cl}(X)]$
If $D > 0$ is a divisor on X then (0.15) says that

$$\dim_k H^0(X, D) = \deg D \tag{0.12}$$

Fix $x \in X$, we define a function

$$\begin{aligned} \theta : X &\rightarrow \mathrm{Cl}^\circ(X) = \{D \in \mathrm{Cl}(X) \mid \deg D = 0\} && (0.13) \\ y &\mapsto y - x && (0.14) \end{aligned}$$

Proposition 0.16
θ is a bijection

Proof
θ is injective: if not, then $\exists f \in k(X)$ s.t. $(f) = y - x$ for some y. This can happen only on $\mathbb{P}_k^1 \Rightarrow$ cannot happen on X

θ is surjective: let's take a divisor D in $\mathrm{Cl}^\circ(X)$, $\deg D + x = \deg D + \deg x = 1$.
So, $\dim_k H^0(X, D + x) = 1$. This means that $\exists f \in k(X)$ s.t. $D' := D + x + (f) \geq 0$.
On the other hand $\deg D' = \deg D + x + (f) = 1$, so D' is a single point, say y.
Therefore, $y = D + x + (f)$ so $D = y - x - (f)$, and hence $D = y - x$ in $\mathrm{Cl}^\circ(X)$ $\square$

1 Sheaf

Definition 1.1
Let X be a topological space. A presheaf $\mathcal{F}$ on X is a collection of data as follows:

(1) $\forall$ open $U \subseteq X$ we have an abelian group $\mathcal{F}(U)$

(2) For any inclusion of open set $V \subseteq U$, we have a "restriction" homomorphism $\mathcal{F}(U) \to \mathcal{F}(V)$ $s \mapsto s|_V$ (Elements of $\mathcal{F}(U)$ are called section)

(3) $\mathcal{F}(\emptyset) = 0$, and the restriction map for $U \subseteq U$ is the indentity map

(4) If $W \subseteq V \subseteq U$ then for any $s \in \mathcal{F}(U), s|_W = (s|_V)|_W$

A sheaf $\mathcal{F}$ on X is a presheaf s.t. it satisfies the following sheaf condition:

$$U = \bigcup_{\text{open}} U_i \quad s_i \in \mathcal{F}(U_i) \text{ s.t. } s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \Rightarrow \exists! s \in \mathcal{F}(U) \text{ s.t. } s|_{U_i} = s_i \tag{1.1}$$

Example 1.2
X topological space. Define $\mathcal{F}(U) = \{s : U \to \mathbb{R} \,|\, s \text{ continuous}\}$. Then $\mathcal{F}$ is a sheaf on X.

(1) $\mathcal{F}(U)$ is an abelian group (of pointwise addition)

(2) the restriction map are given by the usual restriction of functions

(3) $\mathcal{F}(\emptyset) = 0$ (trivial)

(4) trivial

(sheaf condition): if $U = \cup U_i$, $s_i \in \mathcal{F}(U_i)$ s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ then define $s \in \mathcal{F}(U)$ as $s(x) = s_i(x)$ for $x \in U$ if $x \in U_i$

Definition 1.3
$\mathcal{F}$ is a presheaf on a topological space X, and $x \in X$. The stalk of $\mathcal{F}$ at x is defined as

$$\mathcal{F}_x = \varinjlim_{U \ni x} \mathcal{F}(U) \qquad \text{(index over open } U) \tag{1.2}$$

Remark: $\mathcal{F}_x$ is a group with any elements represented by (U, s) where $x \in U$ and $s \in \mathcal{F}(U)$ s.t. $(U, s) = (V, t)$ if there is an open W, $x \in W \subseteq U \cap V$ s.t. $s|_W = t|_W$.

Definition 1.4
X is a topological space, $\mathcal{F}$ and $\mathcal{G}$ presheaves on X. A morphism $\phi : \mathcal{F} \to \mathcal{G}$ is a collection of homomorphism $\mathcal{F}(U) \xrightarrow{\phi_U} \mathcal{G}(U)$ for any open $U \subseteq X$ s.t. if $V \subseteq U$ then the following diagram is commutative

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "\phi_U"] \arrow[d] & \mathcal{G}(U) \arrow[d] \\
\mathcal{F}(V) \arrow[r, "\phi_V"] & \mathcal{G}(V)
\end{tikzcd}$$

Call ϕ an isomorphism if ϕ has an inverse $\psi : \mathcal{G} \to \mathcal{F}$ s.t. $\phi\psi = \iota$

Remark:
(1) If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then a morphism $\phi : \mathcal{F} \to \mathcal{G}$ is just a morphism considering $\mathcal{F}$ and $\mathcal{G}$ as presheaves.

(2) $\phi : \mathcal{F} \to \mathcal{G}$, for any $x \in X$, then we naturally get a homomorphism $\begin{array}{rcl} \phi_x : \mathcal{F}_x & \to & \mathcal{G}_x \\ (U, s) & \mapsto & (U, \phi_U(s)) \end{array}$

Theorem 1.5 (Universal property of sheaf)
X a topological space, $\mathcal{F}$ a presheaf on X. There is a sheaf $\mathcal{F}^+$ and a morphism $\alpha : \mathcal{F} \to \mathcal{F}^+$ given by

$$\forall \text{ open } U \subseteq X, \text{ any } s \in \mathcal{F}(U) \Rightarrow s \in \mathcal{F}^+(U) \text{ where } \begin{array}{rcl} s : U & \to & \bigcup_{x \in U} \mathcal{F}_x \\ x & \mapsto & (U, s) \end{array} \tag{1.3}$$

s.t. for any morphism $\phi : \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ a sheaf, $\exists! \psi : \mathcal{F}^+ \to \mathcal{G}$ s.t. the diagram commutes:

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\alpha"] \arrow[d, "\phi"'] & \mathcal{F}^+ \arrow[dl, "\exists!\,\psi"] \\
\mathcal{G} &
\end{tikzcd}$$

$\mathcal{F}^+$ is called the sheaf associated to $\mathcal{F}$. We can construct $\mathcal{F}^+$ as follows:

$$\mathcal{F}^+(U) = \left\{ s : U \to \bigcup_{x \in U} \mathcal{F}_x \;\middle|\; \begin{array}{l} \forall x \in U \Rightarrow s(x) \in \mathcal{F}_x \\ \forall x \in U \Rightarrow \exists V \ni x, V \subseteq U, \text{ and } t \in \mathcal{F}(V) \text{ s.t. } s(y) = (V, t) \in \mathcal{F}_y \, \forall y \in V \end{array} \right\} \tag{1.4}$$

[Slogan: sections of $\mathcal{F}^+$ are locally sections of $\mathcal{F}$]

Proof
$\mathcal{F}^+$ is a sheaf [exercise]

Existence: Suppose we are given a morphism $\phi : \mathcal{F} \to \mathcal{G}$, define $\psi : \mathcal{F}^+ \to \mathcal{G}$ as follows:
Pick $s \in \mathcal{F}^+(U)$. By sheaf condition, $\exists$ an open cover $\{U_i\}$ (i.e. $U = \cup U_i$) and $s_i \in \mathcal{F}^+(U_i)$ s.t. $s|_{U_i} = s_i$.

By assumptions, we have $t_i = \phi_{U_i}(s_i) \in \mathcal{G}(U_i)$. If $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then $t_i|_{U_i \cap U_j} = \phi_{U_i}(s_i)|_{U_i \cap U_j} = \phi_{U_i \cap U_j}(s_i|_{U_i \cap U_j}) = \phi_{U_i \cap U_j}(s_j|_{U_i \cap U_j}) = \phi_{U_j}(s_j)|_{U_i \cap U_j}$. Therefore by the sheaf condition, $\exists! t \in \mathcal{G}(U)$ s.t. $t|_{U_i} = t_i$. So we can put $\psi|_U(s) = t$

Uniqueness: follows from a theorem that we will prove later. $\square$

Remark: $\forall x \in X$, we have a natural homomorphism $\mathcal{F}_x^+ \to \mathcal{F}_x^{+\prime}$ which is an isomorphism

$$\begin{tikzcd} \mathcal{F}_x \arrow[r] \arrow[dr] & \mathcal{F}_x^+ \arrow[d, dashed, "\cong"] \\ & \mathcal{F}_x^{+\prime} \end{tikzcd}$$

Definition 1.6
X is a topological space, and $\phi : \mathcal{F} \to \mathcal{G}$ a morphism of presheaves. The kernel of ϕ denoted $\ker \phi$ and defined:

$$(\ker \phi)(U) = \ker(\mathcal{F}(U) \xrightarrow{\phi_U} \mathcal{G}(U)) \tag{1.5}$$

($\ker \phi$ is a presheaf)
The image of ϕ denoted by $\operatorname{Im} \phi^{\text{pre}}$ and defined by:

$$\operatorname{Im} \phi^{\text{pre}}(U) = \operatorname{Im}(\mathcal{F}(U) \xrightarrow{\phi_U} \mathcal{G}(U)) \tag{1.6}$$

Proposition 1.7
If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then $\ker \phi$ is a sheaf

Proof
If $U = \bigcup U_i$ is an open cover and $s_i \in (\ker \phi)(U_i) \subseteq \mathcal{F}(U_i)$,then we define

$$s \in (\ker \phi)(U) \text{ s.t. } s|_{U_i} = s_i \tag{1.7}$$

(just apply sheaf condition on $\mathcal{F}$) ($s \in \mathcal{F}(U)$, let $t = \phi_U(s)$: put $t_i = t|_{U_i}$, so $t \in \mathcal{G}(U)$, $t_i = \phi_{U_i}(s_i) = 0$
In other words, t is locally zero and so by the sheaf condition of $\mathcal{G}, t = 0$. So $s \in \ker \phi_U$) $\square$

$\phi : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves on X. Then, $\ker \phi$ is a sheaf but $\operatorname{Im} \phi^{\text{pre}}$ may not be a sheaf. We then define $\operatorname{Im} \phi$ to be the sheaf associated to $\operatorname{Im} \phi^{\text{pre}}$. We call ϕ injective $\ker \phi = 0$ and we call ϕ surjection if $\operatorname{Im} \phi = \mathcal{G}$

Theorem 1.8
$\phi : \mathcal{F} \to \mathcal{G}$ morphism of sheaves on X. Then,

(1) ϕ is injection $\Leftrightarrow \phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is injection $\forall x \in X$

(2) ϕ is surjection $\Leftrightarrow \phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is surjection $\forall x \in X$

(3) ϕ is an isomorphism $\Leftrightarrow \phi_x : \mathcal{F}_x \to \mathcal{G}_x$ is an isomorphism $\forall x \in X$

Proof

(1) ($\Rightarrow$:) Suppose ϕ injective.
Suppose that $(U, \phi_U(s)) = \phi_x(U, s) = 0$ where $x \in U \subseteq X$ and $s \in \mathcal{F}(U)$.
By definition, $\exists W \subseteq U, x \in W$ s.t. $\phi_U(s)|_W = 0$ But we have

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "s \mapsto \phi_U(s)"] \arrow[d, "s \mapsto s|_W"'] & \mathcal{G}(U) \arrow[d, "\phi_U(s) \mapsto 0"] \\
\mathcal{F}(W) \arrow[r] & \mathcal{G}(W)
\end{tikzcd}$$

$\Rightarrow \quad s|_W \in \ker \phi_W$.

$$\text{(So, } \phi \text{ injective } \Rightarrow s|_W = 0 \Rightarrow (U, s) = (W, s|_W) = 0 \\ \Rightarrow \quad \phi_x \text{ injective.)}$$

($\Leftarrow$:) Conversely, assume ϕ_x injective $\forall x \in X$. Take $s \in \mathcal{F}(U)$ with $\phi_U(s) = 0$.
By definition, $\phi_x(U, s) = 0\ \forall x \in U$. So, ϕ_x injective $\Rightarrow \ \phi_x(U, s) = (U, \phi_U(s)) = 0$.
Then $(U, s) = 0$ in $\mathcal{F}_x$. So, theorem is $x \in W \subseteq U$ s.t. $s|_W = 0$. This means that s is locally zero and by the sheaf condition, $s = 0$ hence $\ker \phi_U = 0$

(2) ($\Rightarrow$:) ϕ surjective, and let $x \in X$, $s \in \mathcal{F}$. Pick $(U, t) \in \mathcal{G}_x$.
ϕ surjective $\Rightarrow \ \exists x \in W \subseteq U$ and $s \in \mathcal{F}(W)$ s.t. $t|_W = \phi_W(s)$.
Since $(U, t) = (W, t|_W)$ in $\mathcal{G}_x$ then $\phi_x(W, s) = (W, t|_W)$ hence ϕ_x is surjective.

($\Leftarrow$:) Conversely, assume ϕ_x surjective $\forall x \in X$. Pick $t \in \mathcal{G}(U)$.
Since ϕ_x surjective $\Rightarrow \exists W \subseteq U$, $x \in W$, and $s \in \mathcal{F}(W)$ s.t. $(W, \phi_W(s)) = \phi_x(W, s) = (U, t) \in \mathcal{G}$
So $(W, \phi_W(s)) = (U, t) \ \Rightarrow \ \exists V \subseteq U$ s.t. $\phi_W(s)|_V = t|_V$. So sections of $\mathcal{G}$ are locally the image of section of $\mathcal{F}$ hence ϕ surjective.

(3) Combine above

□

Definition 1.9
X topological space. A <u>sequence of sheaves on X</u> is of the form

$$\cdots \to \mathcal{F}_{-2} \to \mathcal{F}_{-1} \xrightarrow{\phi_{-1}} \mathcal{F}_0 \xrightarrow{\phi_0} \mathcal{F}_1 \xrightarrow{\phi_1} \to \cdots \tag{1.8}$$

where $\mathcal{F}_i$ are sheaves and ϕ_i are morphisms of sheaves. A <u>complex</u> of sheaves is a sequence of sheaves as in equation (**??**) s.t. $\operatorname{Im} \phi_i \subseteq \ker \phi_{i+1}$. Such a complex is an <u>exact sequence</u> if $\operatorname{Im} \phi_i = \ker \phi_{i+1}$. A <u>short exact sequence</u> is a sequence like

$$0 \to \mathcal{F}_1 \hookrightarrow \mathcal{F}_2 \twoheadrightarrow \mathcal{F}_3 \to 0 \tag{1.9}$$

A complex $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is exact iff $0 \to (\mathcal{F}_1)_x \to (\mathcal{F}_2)_x \to (\mathcal{F}_3)_x \to 0$ is exact $\forall x \in X$

Example 1.10 (Constant Sheaf)
X topological space and A an abelian group.
Define $\mathcal{F}^{\text{pre}}$ by $\mathcal{F}^{\text{pre}}(U) = A$ for any open $U \subseteq X$ and take the restriction morphisms to be the identity. $\mathcal{F}^{\text{pre}}$ is a presheaf. Take its associated sheaf $\mathcal{F} = \mathcal{F}^{\text{pre}+}$ which is a sheaf and it is called the constant sheaf defined by A (e.g. $A = \mathbb{Z}$)

We can describe $\mathcal{F}$ as follows: define $\mathcal{G}(U) = \{\text{continuous function } s : U \to A\}$ where A is given the discrete topology. (First prove $\mathcal{G}$ is a sheaf). Define $\phi : \mathcal{F}^{\text{pre}} \to \mathcal{G}$ as follows:

$$\begin{aligned} \phi_U : \mathcal{F}^{\text{pre}}(U) &\to \mathcal{G}(U) \\ s &\mapsto \text{constant function given by } s \end{aligned}$$

Take any $x \in X$. Then, $\mathcal{F}_x = (\mathcal{F}^{\text{pre}})_x = A$

$$\begin{tikzcd} \mathcal{F}^{\text{pre}} \arrow[r, "\phi"] \arrow[dr, "\alpha"'] & \mathcal{G} \arrow[d, "\psi"] \\ & \mathcal{F} \end{tikzcd}$$

On the other hand, $\mathcal{G}_x = A$. Want: $\phi_x : \mathcal{F}_x = A \to \mathcal{G}_x = A$ an isomorphism:
ϕ_x is injective: (i.e. $\phi_x(U, s) = 0$). By considering s a function on U
$\Rightarrow$ $\phi_x(U, s) = 0$ gives an open $W \subseteq U$, $x \in W$ s.t. $s = 0$ on W.

ϕ_x is surjective: pick $(U, t) \in \mathcal{G}_x$ ($x \in U$) and put $s = t(x) \in A$.
Now since A has the discrete topology, $\{s\}$ is considered as a point in A which is an open set, $\Rightarrow W = t^{-1}\{s\}$ is open subset which contains x

Now $(U, t) = (W, t|_W)$ but $t|_W$ is a constant function on W given by s. So, $(U, t) = (W, t|_W) = \phi_x(W, s)$ (this s is considered as an element in $\mathcal{F}^{\text{pre}}(W) = A$) Therefore, $\phi_x = \psi_x$ is an isomorphism $\forall x \in X$, so ϕ is an isomorphism

Definition 1.11
Assume $f : X \to Y$ is an continuous map of topological spaces and $\mathcal{F}$ a sheaf on X. Define direct image of $\mathcal{F}$, $f_*\mathcal{F}$ with $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}U)$ (open $U \subseteq Y$). Then, $f_*\mathcal{F}$ is a sheaf.

If $X = \{y\}$ and f is the inclusion of y in Y, and if $\mathcal{F}$ is a sheaf on X, then $\mathcal{F}$ is completely determined by $A := \mathcal{F}(X)$. In this case,

$$f_*\mathcal{F}(U) = \begin{cases} A & \text{if } y \in U \\ 0 & \text{if } y \notin U \end{cases} \tag{1.10}$$

such a sheaf $f_*\mathcal{F}$ on Y is called the skyscraper sheaf on Y at y given by A

Exercise:$(f_*\mathcal{F})_{y'} = 0$ if $y' \notin \bar{y}$

2 Schemes

Conventions:
All the rings will be commutative with element $1 \neq 0$
A homomorphism $f : A \to B$, we assume $f(1) = 2$
A local ring is ring with a unique maximal ideal. If homomorphism $f : A \to B$ of local rings, we assume that $f(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$ $\qquad (f^{-1}\mathfrak{m}_B \subseteq \mathfrak{m}_A)$

Definition 2.1
A a ring, we define spectrum $\operatorname{Spec}(A)$ to be the set of all prime ideals of A.

If $I \trianglelefteq A$ is any ideal, we define

$$V(I) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid I \subseteq \mathfrak{p}\} \tag{2.1}$$

Lemma 2.2
A is a ring. Then,

(1) $V(IJ) = V(I \cap J) = V(I) \cup V(J)$
(2) $V(\sum I_i) = \cap V(I_i)$
(3) $V(I) \subseteq V(J) \Leftrightarrow \sqrt{I} \supseteq \sqrt{J}$

Proof

(1) $\mathfrak{p} \in V(IJ) \Leftrightarrow IJ \subseteq \mathfrak{p} \Leftrightarrow$ (since $\mathfrak{p}$ prime) $I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p} \Leftrightarrow \mathfrak{p} \in V(I) \cup V(J)$
(similarly for $V(I \cap J)$)

(2) $\mathfrak{p} \in V(\sum I_i) \Leftrightarrow \sum I_i \subseteq \mathfrak{p} \Leftrightarrow I_i \subseteq \mathfrak{p}\ \forall i \Leftrightarrow \mathfrak{p} \in V(I_i)\ \forall i \Leftrightarrow \mathfrak{p} \in \cap V(I_i)$

(3) $V(I) \subseteq V(J) \Leftrightarrow$ every prime ideal $\mathfrak{p}$ that contains I also contains J
Basic fact (from commutative algebra): $\sqrt{I} = \bigcap_{\mathfrak{p} \in V(I)} \mathfrak{p}, \qquad \sqrt{J} = \bigcap_{\mathfrak{p} \in V(J)} \mathfrak{p}$

Then, $\sqrt{I} \supseteq \sqrt{J}$ immediately follows

$\square$

Remark: Taking $V(I)$ to be closed set makes $\mathrm{Spec}(A)$ into a topological space

Definition 2.3
A a ring, $X = \mathrm{Spec}(A)$. We define a sheaf $\mathcal{O}_X$ on X as follows:

$$\mathcal{O}_X(U) = \left\{ s : U \to \bigcup_{\mathfrak{p} \in U} A_{\mathfrak{p}} \;\middle|\; \begin{array}{ll} (1) & \forall\, \mathfrak{p} \in U \Rightarrow s(\mathfrak{p}) \in A_{\mathfrak{p}} \\ (2) & \exists W \subseteq U, \mathfrak{p} \in W,\ a, b \in A \text{ s.t. } s(\mathfrak{q}) = a/b \in A_{\mathfrak{q}}\ \forall\, \mathfrak{q} \in W (b \notin \mathfrak{q}) \end{array} \right\} \tag{2.2}$$

$\mathcal{O}_X$ is a sheaf on X. Every $\mathcal{O}_X(U)$ is actually a ring and all the restriction maps are homomorphism of rings (i.e. $\mathcal{O}_X$ is a sheaf of rings).

Theorem 2.4
A a ring. $X = \mathrm{Spec}(A)$. Then

(1) For any $\mathfrak{p} \in X$, $(\mathcal{O}_X)_{\mathfrak{p}} = A_{\mathfrak{p}}$
(2) For any $b \in A$, $\mathcal{O}_X(D(b)) = A_b$
(3) $\mathcal{O}_X(X) = A$

where $D(b) = X \setminus V(\langle b \rangle)$.
(Note that, for any open set $U \subseteq X, \exists I \trianglelefteq A$ s.t. $U = X \setminus V(I)$.
$\{e_i\}$ generate $I \Rightarrow V(I) = V(\sum \langle e_i \rangle) = \bigcap V(\langle e_i \rangle) \Rightarrow U = \bigcup D(e_i)$)

Proof

(1) We define a homomorphism $f : (\mathcal{O}_X)_{\mathfrak{p}} \to A_{\mathfrak{p}}$ as follows: $(U, s) \mapsto s(\mathfrak{p}) \in A_{\mathfrak{p}}$ this is well-defined because if $(U, s) = (W, t)$ then $s(\mathfrak{p}) = t(\mathfrak{p})$

<u>f is injective:</u>
<u>$f((U, s)) = 0$</u> $\Rightarrow$ s is locally near $\mathfrak{p}$ given by a certain $\frac{a}{b}$ when $a, b \in A \Rightarrow \frac{a}{b} = 0$ in $A_{\mathfrak{p}}$.
By definition of $A_{\mathfrak{p}}$, $\qquad \exists c \in A,\ c \notin \mathfrak{p}$ s.t. $ca = 0$.
Actually, $\frac{a}{b} = 0$ in any point of $D(b) \cap D(c) \supseteq \{\mathfrak{p}\}$.
$\Rightarrow \quad s|_{U \cap D(b) \cap D(c)} = 0$, but $\mathfrak{p} \in U \cap D(b) \cap D(c) \quad \Rightarrow \quad (U, s) = 0$ in $(\mathcal{O}_X)_{\mathfrak{p}}$

<u>f is surjective:</u>
Any $\frac{a}{b} \in A_{\mathfrak{p}}$ defines a function $s : W \to \bigcup_{\mathfrak{q} \in W} A_{\mathfrak{q}}$ given by $\frac{a}{b}$ for some open set $W (\ni \mathfrak{p}) \subseteq X$

$\Rightarrow \quad f((W, s)) = \frac{a}{b} \in A_{\mathfrak{p}}$

(2) (Goal: Know what is $\mathcal{O}_X(D(b))$)

Define $\begin{array}{rcl} g : A_b & \to & \mathcal{O}_X(D(b)) \\ \frac{a}{b^n} & \mapsto & \text{the function given by } \frac{a}{b^n} \end{array}$

<u>g is well-defined:</u>

$$\frac{a}{b^n} = \frac{a'}{b^{n'}} \Rightarrow b^m(ab^{n'} - a'b^n) = 0 \text{ in } A.$$
$$\Rightarrow \frac{a}{b^n} = \frac{a'}{b^{n'}} \in A_{\mathfrak{q}} \text{ for any } \mathfrak{q} \in D(b)$$

<u>g is injective:</u>

Suppose $g(\frac{a}{b^n}) = 0$ in $\mathcal{O}_X(D(b))$, i.e. $\frac{a}{b^n} = 0$ in $A_{\mathfrak{q}} \quad \forall \mathfrak{q} \in D(b)$.
From commutative algebra,
$$A_b = \bigcap_{\mathfrak{q} \in D(b)} A_{\mathfrak{q}}$$
$$\Rightarrow \quad \frac{a}{b} = 0 \text{ in } A_{\mathfrak{q}} \quad \Rightarrow \quad \frac{a}{b} = 0 \text{ in } A_b$$

Elementary way to prove this:
$$\begin{array}{rl} \text{Suppose} & g(\frac{a}{b^n}) = 0 \text{ in } \mathcal{O}_X(D(b)) \\ \Rightarrow & \frac{a}{b^n} = 0 \text{ in } A_{\mathfrak{p}} \quad \forall \mathfrak{p} \in D(b) \\ \Rightarrow & \exists c \notin \mathfrak{p} \text{ s.t. } ca = 0 \end{array}$$
In particular,
$$\begin{array}{rl} & c \in \mathrm{Ann}(a) \\ \Rightarrow & \mathfrak{p} \notin V(\mathrm{Ann}(a)) \quad \text{i.e. } V(\mathrm{Ann}(a)) \cap D(b) = \emptyset \\ \Rightarrow & V(\mathrm{Ann}(a)) \subseteq V(\langle b \rangle) \\ & \Leftrightarrow \sqrt{\mathrm{Ann}(a)} \supseteq \sqrt{\langle b \rangle} \\ & \Leftrightarrow b^r \in \mathrm{Ann}(a) \text{ for some } r \in \mathbb{N}. \\ & \Leftrightarrow \frac{a}{b^n} = 0 \text{ in } A_b. \end{array}$$

<u>g is surjective:</u>

Assume that $s \in \mathcal{O}_X(D(b))$.
By definition of $\mathcal{O}_X$, $\exists$ an open cover $\{U_i\}$ of $D(b)$, and $\frac{a_i}{e_i}$ s.t. $s|_{U_i}$ is given by $\frac{a_i}{e_i}$
Each $U_i = \bigcup_* D(*)$
$\Rightarrow$ can replace U_i by some $D(d_i)$
In particular,
$$\begin{array}{rl} & D(d_i) \subseteq D(e_i) \\ \Rightarrow & V(\langle d_i \rangle) \supseteq V(\langle e_i \rangle) \\ \Rightarrow & \sqrt{\langle d_i \rangle} \subseteq \sqrt{\langle e_i \rangle} \\ \Rightarrow & d_i^{n_i} = {e'}_i e_i \quad \Rightarrow \quad \dfrac{d_i}{e_i} = \dfrac{a_i {e'}_i}{e_i {e'}_i} = \dfrac{a_i {e'}_i}{{d_i}^{n_i}} \end{array}$$
Then, by replacing $\frac{a_i}{e_i}$ by $\frac{a_i {e'}_i}{{d_i}^{n_i}}$ we could assume that s is given by $\frac{a_i}{e_i}$ in $D(e_i)$ (i.e. we have $D(b) = \bigcup D(e_i)$ s.t. $s \in \mathcal{O}_X(D(b))$s.t. $s|_{D(e_i)}$ is given by $\frac{a_i}{e_i}$)

We need only finitely many e_i:
$$\begin{array}{rl} & V(\langle b \rangle) = \bigcap V(\langle e_i \rangle) = V(\sum \langle e_i \rangle) \\ \Rightarrow & \sqrt{\langle b \rangle} = \sqrt{\sum \langle e_i \rangle} \\ \Rightarrow & b^n \in \sum \langle e_i \rangle \text{ for some } n \in \mathbb{N} \end{array}$$
i.e. we can write
$$\begin{array}{rl} & b^n = \sum_i^m l_i e_i \text{ where } l_i \in A \text{ and } m \in \mathbb{N} \\ \Rightarrow & V(\langle b \rangle) \supseteq V(\sum_1^m \langle e_i \rangle) = V(\langle b \rangle) \end{array}$$

Thus,
$$V(\langle b \rangle) = V(\textstyle\sum_1^m \langle e_i \rangle)$$
$$D(b) = \textstyle\bigcup_1^m D(e_i)$$
So we may assume that $e_1, \ldots, e_m$ are all the e_i.

By construction, $s|_{D(e_i)}$ is given by $\frac{a_i}{e_i}$ and so,

$$\left.\frac{a_i}{e_i}\right|_{D(e_i)\cap D(e_j)} = \left.\frac{a_j}{e_j}\right|_{D(e_i)\cap D(e_j)} \qquad \forall i,j \tag{2.3}$$

But $D(e_i)\cap D(e_j) = D(e_ie_j)$, so equation (**??**) $\Rightarrow$ $\frac{a_i}{e_i} = \frac{a_j}{e_j}$ in Ae_ie_j

$\Rightarrow \quad \exists n' \in \mathbb{N}$ s.t. $(e_ie_j)^{n'}(a_ie_j - a_je_i) = 0 \quad \forall i,j$

We can write $\quad e_i^{n'}a_ie_i^{n'+1} - e_j^{n'}a_je_j^{n'+1} = 0$

So can replace $\quad \dfrac{a_i}{e_i}$ by $\dfrac{a_ie_i^{n'}}{e_i^{n'+1}}$ and $\dfrac{a_j}{e_j}$ by $\dfrac{a_je_j^{n'}}{e_j^{n'+1}}$

So the we may assume that $\quad a_ie_j = a_je_i$

Now remember that $\quad b^n = \sum_1^m l_ie_i$

So $\quad e_j(\sum l_ia_i) = \sum l_ie_ja_i = \sum l_ia_je_i = a_j(\sum l_ie_i) = a_jb^n$

$$\Rightarrow \quad \frac{a_j}{e_j} = \frac{\sum l_ia_i}{b^n} \in A_b \quad \Rightarrow \quad g\left(\frac{\sum l_ia_i}{b^n}\right) = s$$

(3) follows form (2) by taking $b = 1$. In this case $D(1) = X$

□

Definition 2.5
A ringed space $(X, \mathcal{O}_X)$ is a topological space X with a sheaf of rings (the structure sheaf) $\mathcal{O}_X$
A morphism $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ between ringed space consists of a continuous map $f : Y \to X$ and a morphism of sheaves $\phi : \mathcal{O}_X \to f_*\,\mathcal{O}_Y$
A locally ringed space is just a ringed space $(X, \mathcal{O}_X)$ s.t. the stalks $(\mathcal{O}_X)_p$ are local ring $\forall p \in X$

A morphism of locally ringed spaces $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a morphism s.t. the induced homomorphism $(\mathcal{O}_X)_\mathfrak{p} \to (\mathcal{O}_Y)_\mathfrak{q}$ (where $f(\mathfrak{q}) = \mathfrak{p} \in X$) is a local homomorphism (i.e. $\phi_x(\mathfrak{m}_{Y,\mathfrak{q}}) \subseteq \mathfrak{m}_{X,\mathfrak{p}}$ and image of 1 is 1).
An isomorphism is a morphism which has an inverse.

Theorem 2.6
Let A, B be rings. Then,

(1) $(X = \operatorname{Spec} A, \mathcal{O}_X)$ is a locally ringed space

(2) Any homomorphism $\alpha : A \to B$ induces (naturally) a morphism of locally ringed spaces:

$$(Y = \operatorname{Spec} B, \mathcal{O}_Y) \to (X = \operatorname{Spec} A, \mathcal{O}_X) \tag{2.4}$$

(with $f : Y \to X \quad , \quad \phi : \mathcal{O}_X \to f_*(\mathcal{O}_Y)$)

(3) Any morphism $(Y = \operatorname{Spec} B, \mathcal{O}_Y) \to (X = \operatorname{Spec} A, \mathcal{O}_X)$ of locally ringed spaces comes from $\alpha : A \to B$ as in (2).

Proof

(1) By 2.4(1), $\quad (\mathcal{O}_X)_p = A_p \quad \Rightarrow \quad (X, \mathcal{O}_X)$ is a locally ringed space.

(2) We define f as: $\quad f(p) = \alpha^{-1}p \in X = \operatorname{Spec} A$.
f is continuous because $f^{-1}(V(I)) = V(\alpha(I)B)$ ✓

We define ϕ as: $\quad$ If $f(p) = q$, then $\exists$local homomorphism $\alpha_p : A_q \to B_p$.
Remember that any $s \in \mathcal{O}_X(U)$ is a function $s : U \to \bigcup_{q \in U} A_q$ (with conditions...).
We define $\quad t \in f_* \mathcal{O}_X(U) = \mathcal{O}_Y(f^{-1}U)$ as a function

$$t : f^{-1}U \to \bigcup_{p \in f^{-1}U} B_p \qquad \text{where } t(p) = \alpha_p(s(f(p))) \in B_p.$$

Locally, s is given by $\frac{a}{b}$ $\Rightarrow$ t is locally given by $\frac{\alpha(a)}{\alpha(b)}$ $\Rightarrow$ $t \in f_* \mathcal{O}_Y(U)$
The induced homomorphism $(\mathcal{O}_X)_q \to (\mathcal{O}_Y)_p$ is nothing but $\alpha_p : A_q \to B_p$ which is a local homomorphism. ✓

(3) Assume that we are given a morphism $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ of locally ringed space $\begin{pmatrix} \bar{f} : Y \to X \\ \bar{\phi} : \mathcal{O}_X \to \bar{f}_* \mathcal{O}_Y \end{pmatrix}$.
$\bar{\phi}$ gives a homomorphism of rings $\mathcal{O}_X(X) \to (f_* \mathcal{O}_Y)(X) = \mathcal{O}_Y(Y)$ which is a homomorphism $\alpha : A \to B$ by 2.4(3).
For $p \in Y$, put $q = \bar{f}(p)$. Then we have a commutative diagram:

$$\begin{tikzcd} A = \mathcal{O}_X(X) \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \mathcal{O}_Y(Y) = B \arrow[d, "\gamma"] \\ A_q = (\mathcal{O}_X)_q \arrow[r, "\bar{\phi}"] & (\mathcal{O}_Y)_p = B_p \end{tikzcd}$$

$$\begin{cases} \alpha^{-1}\gamma^{-1}(p_p) = \alpha^{-1}p \\ \beta^{-1}\bar{\phi}_p^{-1}(p_p) = \beta^{-1}q_q = q \end{cases} \qquad \text{(these 2 lines equal)} \quad \Rightarrow \quad f(p) = q \text{ as in (2).}$$

For any open $U \subseteq X$ we have a diagram (commutative) for $q \in U$

$$\begin{tikzcd} A = \mathcal{O}_X(U) \arrow[r, "\bar{\phi}_U"] \arrow[d, "\beta"'] & \mathcal{O}_Y(f^{-1}U) = B \arrow[d, "f(p) = q"] \\ A_q = (\mathcal{O}_X)_q \arrow[r, "{\alpha_p, \bar{\phi}_p, \phi_p}"] & (\mathcal{O}_Y)_p = B_p \end{tikzcd}$$

s is determined in terms of all the values $s(q)$, $q \in U$.
On the other hand, $\quad \alpha_p(s(q)) = \bar{\phi}_q(s(q)) = \phi_q(s(q))$
But $\phi_U(s)$ and $\bar{\phi}_U(s)$ are determined by the values $\phi_p(s(q))$

$$\Rightarrow \quad \bar{\phi}_U(s) = \phi_U(s) \quad \Rightarrow \quad \bar{\phi} = \phi.$$

□

Definition 2.7
An affine scheme is a locally ringed space which is isomorphic to some $(X = \operatorname{Spec} A, \mathcal{O}_X)$.
A scheme is a locally ringed space $(Y, \mathcal{O}_Y)$ s.t. $\forall y \in Y$ has a neighbourhood U s.t. $(U, \mathcal{O}_X|_U)$ is an affine scheme. (If $\mathcal{F}$ is any sheaf on Y, then $\mathcal{F}|_U$ is the sheaf on U given by $V \subseteq U \mapsto \mathcal{F}|_U(V) = \mathcal{F}(V)$)

- A morphism of schemes is just a morphism of locally ringed spaces
- An isomorphism of scheme is an isomorphism as locally ringed spaces

Example 2.8
A is a ring. Then $X = \operatorname{Spec} A$ is an affine scheme

- A is a field $\Rightarrow$ $X = \{\text{point}\}$
- A is a DVR $\Rightarrow$ $X = \{0, \mathfrak{m}_A\}$, where $\mathfrak{m}_A$ is the maximal ideal.

Here $\mathfrak{m}_A \in X$ is a closed point, i.e. $\{\mathfrak{m}_A\}$ is a closed subset of X; but 0 is not a closed point: actually it is dense, i.e. $\overline{\{0\}} = X$. (0 is generic point)

Example 2.9
A a ring. We say that $\operatorname{Spec} A[t_1, \ldots, t_n]$ is the <u>n-affine space over A</u>, denote $\mathbb{A}^n_A$.

- A is an algebraically closed field $\Rightarrow$ $\mathbb{A}^n_A$ is essentially the classical n-dimensional space.

Note that the current $\mathbb{A}^n_A$ has more points then the classical one.
For example, the points of $\mathbb{A}^1_A$ are the prime ideals $(t-a), a \in A$ plus the zero ideal.

Example 2.10
Take $A = \mathbb{Z}$. Then $\operatorname{Spec} \mathbb{Z} = \{(p) | p \text{ prime or } 0\}$
If $x = 0$, then $\mathcal{O}_x := (\mathcal{O}_X)_x = \mathbb{Z}_0 = \mathbb{Q}$
If $(p) = x \neq 0$, then $\mathcal{O}_x = \mathbb{Z}_{(p)}$ and the <u>residue field</u> $\kappa(x) = \mathcal{O}_x \,/\, \mathfrak{m}_x = \mathbb{F}_p$ (finite field with p element)

Example 2.11
Take the ideal $I = \langle t_1^2 + t_2^2 + 1 \rangle \subseteq \mathbb{R}[t_1, t_2]$.
Classically, $V(I) = \emptyset$. But as a scheme, $X = \operatorname{Spec}(\mathbb{R}[t_1, t_2]/I)$, then $X \neq \emptyset$. We have a homomorphism

$$\mathbb{R}[t_1, t_2]/I \to \mathbb{C}[t_1, t_2]/I \tag{2.5}$$

and this induces a morphism

$$\operatorname{Spec}(\mathbb{C}[t_1, t_2]/I) \to \operatorname{Spec}(\mathbb{R}[t_1, t_2]/I) \tag{2.6}$$

Example 2.12
A a ring, for any ideal $I \trianglelefteq A[t_1, \ldots, t_n]$, we have a scheme $\operatorname{Spec}(A[t_1, \ldots, t_n]/I)$

- If A is an algebraically closed field, then $V(I) \subseteq \mathbb{A}^n_A$ (in the classical sense) defines an affine algebraic set: If J is another ideal with $\sqrt{J} = \sqrt{(I)}$ then $V(I) = V(J)$.

However, as schemes, $\operatorname{Spec}(A[t_1, \ldots, t_n]/I)$ is not necessarily isomorphic to $\operatorname{Spec}(A[t_1, \ldots, t_n]/J)$
For example, $\begin{matrix} I = \langle t \rangle \trianglelefteq A[t] \\ J = \langle t^m \rangle \trianglelefteq A[t] \end{matrix} \quad \Rightarrow \quad \sqrt{I} = \sqrt{J}.$
In an informal sense, $\operatorname{Spec}(A[t]/J)$ correspond to the point $\langle t \rangle \in \mathbb{A}^1_A$ with multiplicity m (see picture)

Definition 2.13
Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring. $\quad \operatorname{Proj} S = \{\mathfrak{p} \in \operatorname{Spec} S \big| \, \mathfrak{p} \text{ homogeneous and } \bigoplus_{d>0} S_d \not\subseteq \mathfrak{p}\}$
$I \trianglelefteq S$ homogeneous ideal $\quad V(I) = \{\mathfrak{p} \in \operatorname{Proj} S | I \subseteq \mathfrak{p}\}$

Lemma 2.14
S a graded ring. Then,

(1) For any homogeneous ideal $I, J \trianglelefteq S$, $V(I) \cup V(J) = V(IJ) = V(I \cap J)$
(2) For any family $\{I_l\}$ of homogeneous ideals in S, we have $\bigcap V(I_l) = V(\sum I_l)$

(Proof as in affine case)

So, by taking $V(I)$ to be closed set, $X = \operatorname{Proj} S$ becomes a topological space.

Definition 2.15
S a graded ring. We define a sheaf of rings $\mathcal{O}_X$ on $X = \operatorname{Proj} S$ as follows:
For any open $U \subseteq X$, let

$$\mathcal{O}_X(U) = \left\{ s : U \to \bigcup_{\mathfrak{p} \in U} S_{(\mathfrak{p})} \,\middle|\, \begin{array}{ll} (1) & \forall \, \mathfrak{p} \in U, s(\mathfrak{p}) \in S_{(\mathfrak{p})} \\ (2) & \exists \text{ open } W \subseteq U, \mathfrak{p} \in W; a, b \in S \text{ homogeneous and same degree} \\ & \quad \text{s.t. } s(\mathfrak{q}) = a/b \in S_{(\mathfrak{q})} \; \forall \, \mathfrak{q} \in W \end{array} \right\} \tag{2.7}$$

So, $(X, \mathcal{O}_X)$ is a ringed space

Theorem 2.16
$S = \bigoplus_{d \geq 0} S_d$ graded ring. Then,

(1) For any $\mathfrak{p} \in X = \operatorname{Proj} S, \mathcal{O}_{\mathfrak{p}} \simeq S_{(\mathfrak{p})}$

(2) For any $b \in \bigoplus_{d>0} S_d$ homogeneous, we have an isomorphism of locally ringed spaces:

$$\left(D_+(b), \mathcal{O}_X \mid_{D_+(b)}\right) \simeq \operatorname{Spec} S_{(b)} \tag{2.8}$$

where $D_+(b) = X \setminus V(\langle b \rangle)$

(3) $(X, \mathcal{O}_X)$ is a scheme.

(Proof not give. Similar to previous cases. (3) follows from (1),(2))

Example 2.17
A any ring, $S = A[t_0, t_1, \dots, t_n]$ is a graded ring. The scheme $\operatorname{Proj} S$ is denoted $\mathbb{P}^n_A$ called projective n-space over A

- A is an algebraically closed field $\Rightarrow$ $\mathbb{P}^n_A$ is essentially the same as the classical n-dimensional projective space over A
- A general. For any homogeneous ideal $I \trianglelefteq S$, we have a corresponding scheme $\operatorname{Proj} S$

Definition 2.18
X is a scheme

- X is <u>reduced</u> if $\mathcal{O}_X(U)$ is reduced (i.e. has no nilpotent elements) for any open $U \subseteq X$
- X is <u>irreducible</u> as a topological space, i.e. if $X = X_1 \cup X_2$ then $X_1 = X$ or $X_2 = X$ (this is not the same as connected)
- X is <u>integral</u> if $\mathcal{O}_X(U)$ is integral (domain) for any $U \subseteq X$

Example 2.19
$X = \operatorname{Spec} A$. Then

- X is reduced $\Leftrightarrow$ A has no nilpotent elements $\Leftrightarrow$ $\operatorname{nil}(A) = \sqrt{0} = 0$
- X is irreducible $\Leftrightarrow$ $\left\{ \begin{array}{c} V(0) = X = V(I) \cup V(J) = V(IJ) \\ \Rightarrow \quad X = V(I) \text{ or } X = V(J) \end{array} \right\}$ $\Leftrightarrow$ $\left\{ \begin{array}{c} IJ \subseteq \sqrt{0} \\ \Rightarrow I \subseteq \sqrt{0} \text{ or } J \subseteq \sqrt{0} \end{array} \right\}$
  $\Leftrightarrow$ $\sqrt{0} = \operatorname{nil}(A)$ is a prime ideal
- X is reduced and irreducible $\Leftrightarrow$ $\operatorname{nil}(A) = 0$ prime ideal $\Leftrightarrow$ $A (= A/0)$ is an integral domain

Lemma 2.20
X a scheme, $s \in \mathcal{O}_X(X)$. Then, the set $W_s = \{x \in X | s \notin \mathfrak{m}_x \subseteq \mathcal{O}_x\}$ is an open subset of X

Proof
By covering X with open affine sets, we may assume that X is affine. Then, for $x \in X$

$s \in \mathfrak{m}_x \subseteq \mathcal{O}_x = A_x \quad \Leftrightarrow \quad s \in x$ (as prime ideal) $\in \operatorname{Spec} A$
(since, from commutative algebra: $\iota_{\mathfrak{p}} : A \to A_{\mathfrak{p}}$, $\iota_{\mathfrak{p}}(\mathfrak{p}) A_{\mathfrak{p}} = \mathfrak{m}_{\mathfrak{p}}$)
$\Leftrightarrow \quad x \in V(\langle s \rangle)$.
$\Leftrightarrow \quad X \setminus W_s$ is closed $\quad \Leftrightarrow \quad W_s$ open. □

Theorem 2.21
A scheme X is integral $\quad \Leftrightarrow \quad$ it is reduced and irreducible

Proof

$\Rightarrow$: Assume that X is integral. Then, $\mathcal{O}_X(U)$ is an integral domain for every open $U \subseteq X$.
$\Rightarrow \quad X$ is reduced by definition.

Suppose X reducible, $X = X_1 \cup X_2$ for proper closed set X_1, X_2
$\Rightarrow \quad (X \setminus X_1) \cap (X \setminus X_2) = \emptyset$ (note $X \setminus X_i \neq \emptyset$ $i = 1, 2$)
Pick open affine set $U_i \subseteq X \setminus X_i$. Sheaf condition for $\mathcal{O}_X \quad \Rightarrow \quad \mathcal{O}_X(U_1 \cup U_2) = \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$
Since $\mathcal{O}_X(U_i) \neq 0 \quad \Rightarrow \quad \mathcal{O}_X(U_1 \cup U_2)$ is not an integral domain #

$\Leftarrow$: Assume that X is reduced and irreducible. Suppose that $\exists U \subseteq X$ and $a, b \in \mathcal{O}_X(U)$ s.t. $ab = 0$ but $a \neq 0, b \neq 0$ in $\mathcal{O}_X(U)$.
By lemma (**??**), $W_a, W_b \subseteq U$ are open sets.
If $V \subseteq U$ is any open affine subset, then $\mathcal{O}_X(V)$ is an integral domain by the previous example.
Now since $a|_V b|_V = 0 \quad \Rightarrow \quad a|_V = 0 \quad$ or $\quad b|_V = 0$.
$\Rightarrow \quad \forall x \in X$
either $\quad a = 0$ in a neighbourhood of x
or $\quad b = 0$ in a neighbourhood of x
$\Rightarrow \quad W_a \cap W_b = \emptyset \quad$ (Otherwise $\exists x \in W_a \cap W_b$ with $a, b \notin \mathfrak{m}_x$. #)
But then $U = (U \setminus W_a) \cup (U \setminus W_b)$ (note that X irreducible $\Rightarrow U$ is irreducible) # (hence X integral)
(if U is not irreducible $\Rightarrow \quad \exists$ open $V, V' \subseteq U$ s.t. $V \cap V' = \emptyset \quad \Rightarrow \quad$ in particular, V, V' are open subsets of X which do not intersect $\quad \Rightarrow \quad X$ not irreducible)

$\square$

Back to the example; $X = \operatorname{Spec} A$ is integral $\quad \Leftrightarrow \quad A$ is an integral domain.

Definition 2.22
X is a scheme. An open subscheme $(U, \mathcal{O}_U)$ is an open set $U \subseteq X$ with structure sheaf $\mathcal{O}_U = \mathcal{O}_X|_U$.
A morphism $f : Y \to X$ is an open immersion if f gives an isomorphism of Y with an open subscheme $U \subseteq X$

Definition 2.23
X a scheme. A closed immersion is a morphism $f : Y \to X$ s.t.

(1) f is a homeomorphism of Y with a closed subset of X
(2) $\mathcal{O}_X \to f_* \mathcal{O}_Y$ is a surjective morphism

A closed subscheme of X is an equivalence class of closed immersion.
Two closed immersions $\begin{array}{c} f : Y \to X \\ f' : Y' \to X \end{array}$ are equivalent if $\exists$ isomorphism $g : Y \to Y'$ and

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "g"'] & X \\ Y' \arrow[ur, "f'"'] & \end{tikzcd}$$

diagram commutes, i.e. $f = f'g$

Example 2.24
$X = \operatorname{Spec} A$ and take $I \trianglelefteq A$ an ideal. Take $Y = \operatorname{Spec} A/I$
$\Rightarrow \quad \exists$ morphism $f : Y \to X$ corresponds to the homomorphism $A \to A/I$
$\Rightarrow \quad f(Y) = V(I)$ and $f : Y \xrightarrow{\text{homeo}} V(I)$
For any $x \in X$ s.t. $x = f(y)$, have homomorphism $\quad A_x = \mathcal{O}_x \twoheadrightarrow \mathcal{O}_y = (A/I)_y = (A/I)_{x/I}$

If $x \notin f(Y)$ then $\mathcal{O}_x \to (f_* \mathcal{O}_Y)_x = 0 \qquad (\mathcal{O}_y = (f_* \mathcal{O}_Y)_x = (\mathcal{O}_Y)_y)$

$\Rightarrow$ the morphism $\mathcal{O}_X \to f_* \mathcal{O}_Y$ surjective
$\Rightarrow$ $\forall n \in \mathbb{N}$, I^n defines a closed immersion $f_n : Y_n = \operatorname{Spec} A/I^n \to X$.
But $f_n(Y_n) = V(I^n) = V(I) \quad \Rightarrow \quad V(I) \subseteq X$ carries infinitely many subscheme structures.
Exercise: Every closed subscheme of X corresponds uniquely to an ideal of A

Definition 2.25
X, Y are scheme over a scheme S: this means that we are given morphisms $f : X \to S$ and $g : Y \to S$. We say that $X \times_S Y$ is the fibred product of X and Y over S if we have a commutative diagram:

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ X \times_S Y \arrow[ur, "p"] \arrow[dr, "q"'] & & S \\ & Y \arrow[ur, "g"'] & \end{tikzcd}$$

s.t. for any other commutative diagram:

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ Z \arrow[ur, "\pi"] \arrow[dr, "\mu"'] & & S \\ & Y \arrow[ur, "g"'] & \end{tikzcd}$$

$\exists!$ morphism $\phi : Z \to X \times_S Y$ giving us a commutative diagram:

$$\begin{tikzcd} Z \arrow[dr, "\varnothing"] \arrow[drr, "\pi"] \arrow[ddr, "\mu"'] & & \\ & X \times_S Y \arrow[r, "p"'] \arrow[d, "q"] & X \arrow[d, "f"] \\ & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Theorem 2.26
X, Y schemes over a scheme S where X, Y, S are affine. Then the fibred product exists for affine schemes

Proof
$X \times_S Y = \operatorname{Spec} B \otimes_A C$. If $Z = \operatorname{Spec} D$ and if we have a commutative diagram on schemes, then we have a commutative diagram on rings:

$$\begin{tikzcd} & X \arrow[dr] & & & & B \arrow[dl] & \\ Z \arrow[ur] \arrow[dr] & & S & \rightsquigarrow & D & & A \arrow[ul] \arrow[dl] \\ & Y \arrow[ur] & & & & C \arrow[ul] & \end{tikzcd}$$

By universal properties of the tensor product we get a commutative diagram on rings which then translate into one on Spec:

$$
\begin{tikzcd}
D & & \\
& B \otimes_A C \arrow[ul] & B \arrow[l] \arrow[ull] \\
& C \arrow[u] \arrow[uul] & A \arrow[l] \arrow[u]
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
Z \arrow[dr] \arrow[drr] \arrow[ddr] & & \\
& X \times_S Y \arrow[r] \arrow[d] & X \arrow[d] \\
& Y \arrow[r] & Z
\end{tikzcd}
$$

The uniqueness is follows from the uniqueness of the tensor product $B \otimes_A C$ $\square$

Remark: In the general case, we need to cover X, Y, S by open affine schemes, construct the fibred products of the affine set and finally glue all of them together to get the product $X \times_S Y$

Definition 2.27
$f : X \to Y$ a morphism of scheme, $y \in Y$. The fibre of f over y is defined as

$$X_y := X \times_Y \operatorname{Spec} k(y) \tag{2.9}$$

where $k(y)$ is the residue field of y

Example 2.28
Let X be a scheme. $\exists!$ morphism $f : X \to \operatorname{Spec} \mathbb{Z}$ which corresponds to $\mathbb{Z} \to \mathcal{O}_X(X)$
Assume $y = (0)$. In this case $k(y) = \mathcal{O}_y / \mathfrak{m}_y = \mathbb{Q}/0 = \mathbb{Q}$.
$\Rightarrow \quad X_y = X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Q} \qquad$ (if $X = \operatorname{Spec} A$, then $X_y = \operatorname{Spec}(A \otimes_{\mathbb{Z}} \mathbb{Q})$)
If $y = (p)$, p prime in $\mathbb{Z}$. The residue field $k(y) = \mathbb{F}_p$ (finite field of size p)
$\Rightarrow \quad X_y = X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{F}_p \qquad$ (if $X = \operatorname{Spec} A$, then $X_y = \operatorname{Spec}(A \otimes_{\mathbb{Z}} \mathbb{F}_p)$)
This is the "reduction mod p" method

Example 2.29
Suppose that X is a scheme over a field K (equivalent to over $\operatorname{Spec} K$) and L/K field extension. The fibred product $X \times_{\operatorname{Spec} K} \operatorname{Spec} L$ is a scheme over the $\operatorname{Spec} L$ (if $X = \mathbb{A}^n_K$, then $X \times_{\operatorname{Spec} K} \operatorname{Spec} L = \mathbb{A}^n_L$)

Example 2.30
Let $f : X \to Y$ be a morphism where Y is integral. Then, Y has a unique generic point η. The generic fibre of f is defined to be X_η. By definition, X_η is a scheme over $\kappa(\eta)$=function field of Y

Definition 2.31
$f : X \to Y$ morphism of schemes, f is called separated if the diagonal morphism $\Delta : X \to X \times_Y X$ $(x \mapsto (x, x))$ is a closed immersion

$$
\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[drr, "\mathrm{id}"] \arrow[ddr, "\mathrm{id}"'] & & \\
& X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& X \arrow[r, "f"] & Y
\end{tikzcd}
$$

Remark. A topological space X is Hausdorff if for any $x \neq x'$, $(x, x' \in X)$, $\exists$ open $W, W' \subseteq X$, containing x, x' respectively, s.t. $W \cap W' = \emptyset$
Fact: $\quad X$ is Hausdorff $\Leftrightarrow$ $\Delta(X) = \{(x, x) \in X \times X | x \in X\} \subseteq X \times X$ is a closed subset.

Example 2.32
K a field, $X = \mathbb{A}^1_K$. Then $X \to \operatorname{Spec} K$ is a separated morphism, due to the following theorem.

Theorem 2.33
Let $f : X \to Y$ be a morphism of affine schemes. Then, f is separated.

Proof
Let $X = \operatorname{Spec} A, Y = \operatorname{Spec} B$. Then the diagram (in the definition **??**) correspond to

$$\begin{tikzcd}
A & & \\
& A \otimes_B A \arrow[ul, "\alpha"'] & A \arrow[l] \arrow[ull, "\mathrm{id}"'] \\
& A \arrow[u] \arrow[uul, "\mathrm{id}"] & B \arrow[l] \arrow[u]
\end{tikzcd}$$

α is given by $a \otimes a' \mapsto aa'$
α is surjective so the corresponding morphism $\Delta : X \to X \times_Y X$ is a closed immersion. $\square$

Theorem 2.34
$f : X \to Y$ is separated $\quad\Leftrightarrow\quad$ $\Delta(X)$ is a closed subset of $X \times_Y X$

Proof
$\Rightarrow :$ If f is separated then Δ is a closed immersion hence $\Delta(X)$ is closed.

$\Leftarrow$: Conversely, suppose that $\Delta(X)$ is closed.
To prove that Δ is a closed immersion we need to check that

- Δ is a homomorphism of X to $\Delta(X)$: by definition of $X \times_Y X$ we have the commutative diagram (from part of the commutative diagram in the definition **??**)

$$\begin{tikzcd}
X \arrow[r, "\Delta"] \arrow[dr, "\mathrm{id}"'] & \Delta(X) \arrow[d, "(x,x) \mapsto x"] \\
& X
\end{tikzcd}$$

  $\Rightarrow \quad \Delta : X \twoheadrightarrow \Delta(X)$
- $\mathcal{O}_{X \times_Y X} \to \Delta_* \mathcal{O}_X$ is surjective: we check this on stalks. Let $q \in X \times_Y X$. First assume that $q \notin \Delta(X)$. Since $\Delta(X)$ is closed, $(\Delta_* \mathcal{O}_X)_q = 0$ hence surjectivity is trivial in this case. (informally, this means $q = (p, p)$)
  Now assume that $q = f(p)$ for some $p \in X$. If U is an open affine subscheme in X containing p and if $V \subseteq Y$ open affine, then $inU \times_V U$ ($f(U) \subseteq V$)
  So we need only to consider $\Delta|_U : U \to U \times_V U$. We know that $\Delta|_U$ is a closed immersion hence

$$\begin{array}{ccc}
(\mathcal{O}_{U \times_V U})_q & \twoheadrightarrow & ((\Delta|_U)_* \mathcal{O}_U)_q \\
\| & & \| \\
(\mathcal{O}_{X \times_Y X})_q & \to & (\Delta_* \mathcal{O}_X)_q
\end{array} \tag{2.10}$$

  (LHS equal because $U \times_V U \subseteq X \times_Y X$ is open containing q)
  Therefore we have surjectivity for the lower map

$\square$

Theorem 2.35 (Valuation Criterion for Separatedness)
Suppose X Noetherian. (i.e. if for any sequence $U_1 \subseteq U_2 \subseteq \cdots$ $\exists n$ s.t. $U_i = U_{i+1}$ $\forall i \geq n$)
$f : X \to Y$ is separated $\Leftrightarrow$ the following holds: for any valuation ring R with fraction field K and any commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

there is at most one g fitting into the diagram
(A local ring R inside a field L is a valuation ring for L if we have any other local ring A with $R \overset{\alpha}{\hookrightarrow} A \hookrightarrow L$ s.t. α is local, then $R = A$)

Example 2.36
Let k be a field, and $U = \mathbb{A}^1_k, V = \mathbb{A}^1_k$. We glue U and V along the subdivisions $U \setminus \{0\}$ and $V \setminus \{0\}$, call it X. Let x, x' be the two origins.
We have a diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] & \operatorname{Spec} K \arrow[l] \arrow[d] \\ \operatorname{Spec} \mathcal{O}_x \arrow[r] \arrow[ur, "g"] & \operatorname{Spec} K & \operatorname{Spec} \mathcal{O}_{x'} \arrow[l] \arrow[ul, "g'"'] \end{tikzcd}$$

where K=function field of X=fraction field of $\mathcal{O}_x$
Now if you ignore g and g', then the two sides are isomorphic.

So, we have two different morphisms form $\operatorname{Spec} R$ into X where $R := \mathcal{O}_x \cong \mathcal{O}_{x'}$ Therefore X is not separated over $\operatorname{Spec} K$ by the theorem

Example 2.37
Let $f : X \to Y$ is a closed immersion. Then f is separated. Suppose that we have a diagram:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} R \arrow[r, "h"] \arrow[ur, "g"] & Y \end{tikzcd}$$

Suppose that $y = h(\mathfrak{m}_R)$, $y' = h(\eta)$
(η is a generic point of $\operatorname{Spec} R$)
Since the diagram is commutative $\Rightarrow$ $y' \in f(X)$
since $f(X)$ is closed in Y $\Rightarrow$ $y \in f(X)$.
Take any open affine $U \subseteq Y$ $(y \in U)$.
$\Rightarrow$ If $V = f^{-1}U$ then we have commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & V \arrow[r, hook] \arrow[d, "\text{closed immersion}"] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r, "h"] & \operatorname{Spec} K \arrow[r, hook] & Y \end{tikzcd}$$

Since $h^{-1}U$ is open $\Rightarrow$ $\eta \in h^{-1}U$ $\Rightarrow$ $y' \in U$
If $U = \operatorname{Spec} A$ then $V = \operatorname{Spec} A/I$ some ideal $I \trianglelefteq A$
Then we have a diagram

$$\begin{tikzcd} K & A/I \arrow[l] \\ R \arrow[u, hook] \arrow[r] & A \arrow[u] \end{tikzcd}$$

since $A \to A/I$ is surjective, there is at most one homomorphism $A/I \to R$ fits into the diagram

Remark. If $f : X \to Y$ is a continuous map of topological space, we say that it is proper if $f^{-1}S$ is compact for any compact $S \subseteq Y$

Definition 2.38
A morphism of schemes $f : X \to Y$ is proper if:

- f is separated
- f is of finite type:

  $\exists$ open affine cover $\bigcup U_i = Y \qquad (U_i = \operatorname{Spec} A_i)$
  s.t. $\forall i\ \exists$ open affine cover $\bigcup V_\alpha = f^{-1}U_i \qquad (V_\alpha = \operatorname{Spec} B_\alpha)$
  s.t. each B_α is finitely generated over A_i
- f is universally closed:
  for any morphism $S \to Y$ and for any closed subset of $S \times_Y X$ the image $p(T)$ is closed

$$\begin{tikzcd}
T \arrow[d] \arrow[r, phantom, "\overset{\text{closed}}{\subseteq}"] & S \times_Y X \arrow[r] \arrow[d, "p"] & X \arrow[d] \\
p(T) \arrow[r, phantom, "\underset{\text{closed}}{\subseteq}"] & S \arrow[r] & Y
\end{tikzcd}$$

Example 2.39
Put $X = \mathbb{A}^1_k$ where k is a field. Then $f : X \to \operatorname{Spec} k$ is not proper.
Take $S = \mathbb{A}^1_k$ and take $T \subseteq \mathbb{A}^1_k \times_k \mathbb{A}^1_k = \mathbb{A}^2_k$ to be $V(t_1t_2 - 1)$. Now the image of T under p is equal to $\mathbb{A}^1_k \setminus \{0\}$ which is not closed

$$\begin{tikzcd}
T \arrow[r, phantom, "\subseteq"] & \mathbb{A}^2_k \arrow[r] \arrow[d, "p"'] & \mathbb{A}^1_k \arrow[d] \\
& \mathbb{A}^1_k \arrow[r] & \operatorname{Spec} K
\end{tikzcd}$$

<u>Important Remark:</u>
A scheme X is Noetherian if X can be covered by finitely many affine $U = \operatorname{Spec} A$ s.t. A is Noetherian. This is a more useful definition than the one given before.

Theorem 2.40 (Valuation Criterion for Properness)
Let $f : X \to Y$ be a morphism with X Noetherian, and f of finite type.
Then f is proper $\Leftrightarrow$ for any valuation ring R with field K and a commutative diagram as below, then there is a unique g fitting into the diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\operatorname{Spec} R \arrow[r, "h"'] \arrow[ur, dashed, "\exists! g"] & Y
\end{tikzcd}$$

Proof
(Sketch)
Let $U = \operatorname{Spec} K$, $Z = \operatorname{Spec} R$.
Suppose that f is proper and a diagram as in the theorem. Since f is separated, $\exists$ at most one g.

Take the fibred product

$$\begin{tikzcd}
U \arrow[dr, "h"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& Z \times_Y X \arrow[r] \arrow[d, "p"] & X \arrow[d] \\
& Z \arrow[r] & Y
\end{tikzcd}$$

Put $a = h(U)$ and put $T = \overline{\{a\}}$. Take a point $b \in T$ s.t. $(b) = \mathfrak{m}_R$ ($\exists$ such a b since $p(T)$ is closed) We take an integral scheme structure on T. Now we have a local homomorphism of local rings $(\mathcal{O}_Z)_{\mathfrak{m}_R} = R \to (\mathcal{O}_T)_b$. Now a is the generic point of T. We have also a homomorphism $\kappa(a) \hookrightarrow K$. On the other hand, $(\mathcal{O}_T)_b \hookrightarrow \kappa(a)$ and so we get $R \hookrightarrow (\mathcal{O}_T)_b \hookrightarrow \kappa(a) \hookrightarrow K$. This situation induces a morphism $Z \to T$ and so a morphism $g : Z \to X$ which fits into the diagram

To prove that f is proper are only to verify that f is unversally close:
f is of finite type by assumption, and f is separated by theorem **??**

Suppose that we are given a morphism $S \to Y$ and suppose that $T \subseteq S \times_Y X$ is closed. We put the integral scheme structure on T

$$\begin{tikzcd}
T \arrow[r, hook] \arrow[dr] & S \times_Y X \arrow[r] \arrow[d, "p"] & X \arrow[d, "f"] \\
& S \arrow[r] & Y
\end{tikzcd}$$

It turns out that to prove that $p(T)$ is closed, it is enough to prove that $\overline{\{a\}} \subseteq p(T)$ if $a \in p(T)$
Pick $a \in p(T)$, $p(T) = a$ for some $t \in T$, and let $S' = \overline{\{a\}}$ and $b \in S'$. We put the integral scheme structure on S'. We get a morphism $\operatorname{Spec}(\mathcal{O}_{S'})_b \to S'$
On the other hand we have homomorphism $(\mathcal{O}_{S'})_b \hookrightarrow \kappa(a) \hookrightarrow \kappa(t)$
There is a valuation ring R for $\kappa(t)$ s.t. $(\mathcal{O}_{S'})_b \to R$ is a local homomorphism. So we get a diagram

$$\begin{tikzcd}
\operatorname{Spec} R \arrow[drr, "h"] \arrow[drrr, dashed, "g"] \arrow[dd, bend right] & & & \\
\operatorname{Spec} \kappa(t) \arrow[u] \arrow[r] & T \arrow[r, hook] & S \times_Y X \arrow[r] \arrow[d, "p"] & X \arrow[d, "f"] \\
\operatorname{Spec}(\mathcal{O}_{S'})_b \arrow[r] & S' \arrow[r] & S \arrow[r] & Y
\end{tikzcd}$$

By the valuation criterion $\exists! g$ which fits into the diagram since $S \times_Y X$ is a fibred product, g determines h uniquely.

Since generic point of $\operatorname{Spec} R$ goes into T and T is closed, so $h(\operatorname{Spec} R) \subseteq T$.
In particular, $\exists t'$ s.t. $h(\mathfrak{m}_R) = t'$
Since the diagram is commutative $p(t') = b$, so $b \in p(T)$. Therefore $p(T)$ is closed in S $\square$

Example 2.41
<u>Claim</u>:
With Noetherian schemes, we have the following:

(1) a closed immersion is proper
(2) a composition of proper morphisms is again proper

Proof

(1) Let $f : X \to Y$ be a closed immersion.
If $U = \operatorname{Spec} A \subseteq Y$, then $f^{-1}U = \operatorname{Spec} A/I$ for some ideal $I \trianglelefteq A$
Now $A \to A/I$ is of finite type, i.e. A/I is a finitely generated A-algebra
To use the valuation criterion proceed exactly as in example **??**

(2) Let $f : X \to Y$, $g : Y \to Z$ be proper morphisms. The composition gf is of finite type so we can use the valuation criterion. So that we have commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d, "gf"] & Y \arrow[dl, "g"] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "e"] & Z &
\end{tikzcd}$$

So we have a map $\operatorname{Spec} K \to Y$
Since g is proper we get a unique $h : \operatorname{Spec} R \to Y$
Now since f is proper we then get a unique morphism e. Therefore, gf is proper.

□

Definition 2.42
Y a scheme. We define the projection n-space

$$\mathbb{P}^n_Y := Y \times_{\operatorname{Spec} \mathbb{Z}} \mathbb{P}^n_{\mathbb{Z}}$$

Morphism $f : X \to Y$ is called projective if it can be factored as

$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[d, "f"] & \mathbb{P}^n_Y \arrow[dl, "p"] \\
Y &
\end{tikzcd}$$

where g is a closed immersion.
A quasi-projective morphism is a composition of an open immersion followed by a projective morphism

Theorem 2.43
Any projective morphism of Noetherian schemes is proper

Proof
Suppose that $f : X \to Y$ is a projective morphism of Noetherian schemes. f is of finite type since it is a composition of closed immersion and some $\mathbb{P}^n_Y \xrightarrow{p} Y$
Now suppose that we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[r, "g"] \arrow[d, "f"'] & \mathbb{P}^n_Y \arrow[r] \arrow[dl, "p"] & \mathbb{P}^n_{\mathbb{Z}} \arrow[dl, "q"] \\
\operatorname{Spec} R \arrow[r] & Y \arrow[r] & \operatorname{Spec} \mathbb{Z} &
\end{tikzcd}$$

It is enough to prove that p is proper because g is a closed immersion and proper (by Example **??**)
By the universal property of fibred products, it is enough to prove that q is proper. So we then assume that we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "e"] \arrow[d] & \mathbb{P}^n_{\mathbb{Z}} \arrow[d, "q"] \\
\operatorname{Spec} R \arrow[r] & \operatorname{Spec} \mathbb{Z}
\end{tikzcd}$$

Let $a = e(\operatorname{Spec} K)$.
Now by Theorem 2.16, $\mathbb{P}^n_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[t_0, \dots, t_n]$ is covered by

$$U_i = \operatorname{Spec} \mathbb{Z}[t_0, \dots, t_n]_{(t_i)} = D_+(t_i) \qquad (\mathbb{Z}[t_0, \dots, t_n]_{(t_i)} = \mathbb{Z}[t_0/t_i, \dots, t_n/t_i])$$

If $a \notin U_i \quad \Rightarrow \quad a$ belongs to the closed subscheme given by t_i which is just $\mathbb{P}^{n-1}_{\mathbb{Z}}$ so we can do induction (see diagram)

If $a \in U_i \ \forall i \quad \Rightarrow \quad t_j/t_i \in \mathcal{O}_a \ \forall j$.
In particular, since $t_i/t_j \in \mathcal{O}_a$, these are units in $\mathcal{O}_a$
We have homomorphisms:

$$\begin{tikzcd}
\mathbb{Z}[t_0/t_i, \dots, t_n/t_i] \arrow[r, "\beta"] & \mathcal{O}_a \arrow[r, "\alpha"] & K \arrow[r, "v"] & G \\
& \mathbb{Z} \arrow[r] \arrow[ul] & R \arrow[u, hook] \arrow[ur] &
\end{tikzcd}$$

Let $s_{ji} = (\alpha\beta)(t_j/t_i)$
First note that $s_{ji}s_{il} = s_{jl}$
Fact: $\exists$ a valuation v where G is totally ordered abelian group:

$$\begin{cases} v(\mu\theta) = v(\mu) + v(\theta) \\ v(\mu + \theta) \geq \min\{v(\mu), v(\theta)\} \end{cases}$$

$R = \{\mu \in K | v(\mu) \geq 0\} \qquad \mathfrak{m}_R = \{\mu \in K | v(\mu) > 0\}$

Note that $v(s_{jl}) = v(s_{ji}) + v(s_{il})$
Choose j s.t. $v(s_{j0}) = \min\{v(s_{10}), \dots, v(s_{n0})\}$
$\Rightarrow \quad \forall i \quad v(s_{ij}) = v(s_{i0}) - v(s_{j0}) \geq 0$
$\Rightarrow \quad s_{ij} \in R$
So we get a natural homomorphism $\mathbb{Z}[t_0/t_j, \dots, t_n/t_j] \to R$ which gives a morphism $\operatorname{Spec} R \to U_j \subseteq \mathbb{P}^n_{\mathbb{Z}}$
$\exists$ at most one morphism $\operatorname{Spec} R \to \mathbb{P}^n_{\mathbb{Z}}$, otherwise the image of $\operatorname{Spec} R$ would be in another U_i because this forces $(\alpha\beta)(t_j/t_i) \in R$ which contradicts the fact $(\alpha\beta)(t_i/t_j) \in R$ and local homomorphism given by $\operatorname{Spec} R \to \mathbb{P}^n_{\mathbb{Z}}$ $\square$

3 $\mathcal{O}_X$-modules

Definition 3.1
$(X, \mathcal{O}_X)$ a ringed space. An $\underline{\mathcal{O}_X\text{-module}}$ is a sheaf $\mathcal{F}$ on X s.t. for any open $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module s.t. if $V \subseteq U$, then if $s \in \mathcal{O}_X(U)$ and $m \in \mathcal{F}(U)$ then $(sm)|_V = (s|_V)(m|_V)$

- A <u>morphism</u> $\phi : \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules is a morphism of sheaves s.t. for any open $U \subseteq X$, $\mathcal{F}(U) \to \mathcal{G}(U)$ must be an $\mathcal{O}_X(U)$-homomorphism
- The <u>direct sum</u>, <u>direct limit</u>, <u>inverse limit</u> of $\mathcal{O}_X$-modules are again $\mathcal{O}_X$-module
- The <u>kernel</u> and the <u>image</u> of a morphism of $\mathcal{O}_X$-modules are again $\mathcal{O}_X$-module
- $\mathcal{F}, \mathcal{G}$ are $\mathcal{O}_X$-modules
  Define $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ to be the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X} \mathcal{G}(U)$
- $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces and $\mathcal{F}$ an $\mathcal{O}_X$-modules
  $\Rightarrow \quad f_* \mathcal{F}$ is an $\mathcal{O}_Y$-module (remember $\mathcal{O}_Y \to f_* \mathcal{O}_X$)

Definition 3.2
$X = \operatorname{Spec} A$, M an A-module. For any open $U \subseteq X$ let

$$\widetilde{M}(U) = \left\{ s : U \to \bigcup_{\mathfrak{p} \in U} M_{\mathfrak{p}} \,\middle|\, \begin{array}{c} \forall \mathfrak{p} \in U, s(\mathfrak{p}) \in M_{\mathfrak{p}} \\ \exists \text{ open } W \subseteq U, W \ni \mathfrak{p} \text{ and } m \in M, a \in A \\ \text{s.t. } s(\mathfrak{q}) = \frac{m}{\alpha} \in M_{\mathfrak{p}} \;\; \forall \mathfrak{q} \in W, \alpha \notin \mathfrak{q} \end{array} \right\} \tag{3.1}$$

$\widetilde{M}$ is a sheaf. Actually it is an $\mathcal{O}_X$-module:
For open $U \subseteq X$, $b \in \mathcal{O}_X(U)$, $s \in \widetilde{M}(U)$, we put $bs \in \widetilde{M}(U)$ to be the function

$$bs : U \to \bigcup_{\mathfrak{p} \in U} M_{\mathfrak{p}} \tag{3.2}$$

$$(bs)(\mathfrak{p}) = b(\mathfrak{p}) s(\mathfrak{p}) \qquad \text{(where } b(\mathfrak{p}) \text{ is the image of } b \text{ in } A_{\mathfrak{p}}) \tag{3.3}$$

If locally $s = m/a$, then $bs = bm/a$ locally

Theorem 3.3
$X = \operatorname{Spec} A$, M an A-module. Then

(1) $\widetilde{M}$ is an $\mathcal{O}_X$-module

(2) For any $\mathfrak{p} \in X$, $\widetilde{M}_{\mathfrak{p}} = M_{\mathfrak{p}}$

(3) For any $b \in A$, $\widetilde{M}(D(b)) \cong M_b$

(4) $\widetilde{M}(X) \cong M$

Proof
(1) was discussed above.
(2),(3),(4) are similar to the proof of Theorem 2.4 □

Definition 3.4
$f : X \to Y$ is a continuous map of topological map and $\mathcal{G}$ a sheaf on Y. For every open set $U \subseteq X$, put

$$\mathcal{P}(U) = \varinjlim_{f(U) \subseteq V} \mathcal{G}(V) \tag{3.4}$$

We define the image of $\mathcal{G}$ denoted by $f^{-1}\mathcal{G}$ to be the sheaf associated to $\mathcal{P}$

(Idea: given $A \to B$, and M an A-module, to construct a B-module, we use $M \otimes_A B$)

- If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, and $\mathcal{G}$ an $\mathcal{O}_Y$-module, then we define

$$f^* \mathcal{G} = f^{-1} \mathcal{G} \otimes_{f^{-1} \mathcal{O}_Y} \mathcal{O}_X \tag{3.5}$$

We have a morphism $\mathcal{O}_Y \to f_* \mathcal{O}_X$
$\Rightarrow$ get a morphism $f^{-1} \mathcal{O}_Y \to f^{-1} f_* \mathcal{O}_X$
We also have a natural morphism $f^{-1} f_* \mathcal{O}_X \to \mathcal{O}_X$
$\Rightarrow$ Compose the two $f^{-1} \mathcal{O}_Y \to \mathcal{O}_X$.
On the other hand $f^{-1} \mathcal{G}$ is an $f^{-1} \mathcal{O}_Y$-module, so we can form $f^{-1} \mathcal{G} \otimes_{f^{-1} \mathcal{O}_Y} \mathcal{O}_X$

$$\begin{tikzcd}
f^* \mathcal{G} = A \otimes_R B & f^{-1} \mathcal{G} = B \arrow[l] \\
A = \mathcal{O}_X \arrow[u] & f^{-1} \mathcal{O}_Y = R \arrow[l] \arrow[u]
\end{tikzcd}$$

Theorem 3.5
$X = \operatorname{Spec} A$. Then,

(1) For any A-modules M, N we have $\widetilde{M \otimes_A N} \cong \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}$

(2) For any family of A-module $\{M_i\}$ we have $\widetilde{\bigoplus M_i} \cong \bigoplus \widetilde{M_i}$

(3) If $X = \operatorname{Spec} A \xrightarrow{f} \operatorname{Spec} B = Y$ and M an A-module, then $f_* \widetilde{M} \cong \widetilde{{}_B M}$ (${}_B M$ is just M considered as a module over B)

(4) Assumption as in (3), if N is a B-module, then $f^* \widetilde{N} \cong \widetilde{N \otimes_B A}$

Proof

(1) For any open $U \subseteq X$ and $s \in \widetilde{M}(U)$, $t \in \widetilde{N}(U)$
Define $r \in \widetilde{M \otimes_A N}(U)$ as follows:

$$\begin{aligned} r : U &\to \bigcup (M \otimes_A N)_{\mathfrak{p}} \cong M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} && (3.6) \\ r(\mathfrak{p}) &= s(\mathfrak{p}) \otimes t(\mathfrak{p}) && (3.7) \end{aligned}$$

If locally $s = m/a$ and $t = n/b$, then $r = \frac{m \otimes n}{ab}$ Note that the function $(s, t) \mapsto r$ is bilinear hence we get a homomorphism $\widetilde{M}(U) \otimes_{\mathcal{O}_X(U)} \widetilde{N}(U) \to \widetilde{M \otimes N}(U)$
So if $\mathcal{P}$ is the presheaf given by $\mathcal{P}(U) = \widetilde{M}(U) \otimes_{\mathcal{O}_X(U)} \widetilde{N}(U)$, then we have a morphism

$$\phi : \mathcal{P} \to \widetilde{M \otimes N} \tag{3.8}$$

which determines a morphism

$$\phi^+ : \mathcal{P}^+ = \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \to \widetilde{M \otimes_A N} \tag{3.9}$$

For any $\mathfrak{p} \in X$, the homomorphism $\phi^+_{\mathfrak{p}} = \phi_{\mathfrak{p}} = \mathcal{P}^+_{\mathfrak{p}} = \mathcal{P}_{\mathfrak{p}} \to (\widetilde{M \otimes_A N})_{\mathfrak{p}} \cong (M \otimes_A N)_{\mathfrak{p}}$ which is an isomorphism.
Then ϕ^+ is an isomorphism (Exercise (on Example Sheet 3): $\mathcal{P}^+_{\mathfrak{p}} = M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}}$)

(2) Proof as in (1) but easier

(3) Define a morphism from $\psi \widetilde{{}_B M} \to f_* \widetilde{M}$ as follows:
For any open $U \subseteq X$ and $s \in \widetilde{{}_B M}(U)$ then put

$$\psi_U(s) : f^{-1} U \to \bigcup_{\mathfrak{p} \in U} M_{\mathfrak{p}} \tag{3.10}$$

s is a function $U \to \bigcup_{\mathfrak{q}\in U} ({}_BM)_{\mathfrak{q}}$ and if $f(\mathfrak{p}) = \mathfrak{q}$ then we have a homomorphism $\beta_{\mathfrak{p}} : {}_BM_{\mathfrak{q}} \to M_{\mathfrak{p}}$.
Now put $\psi_U(s)(\mathfrak{p}) = \beta_{\mathfrak{p}} s(f(\mathfrak{p})) \in M_{\mathfrak{p}}$
If s is locally given by a certain m/b where $m \in {}_BM$ and $b \in B$. Then locally $\psi_U(S)$ is given by $m/\alpha(b)$ where $\alpha : B \to A$ is the homomorphism corresponding to f
Observe that if $b \in B$ then $f^{-1}D(b) = D(\alpha(b))$. Now ψ_U is

$$\widetilde{{}_BM}(D(b)) \cong {}_BM_b \to (f_*\widetilde{M})(D(b)) = \widetilde{M}(f^{-1}D(b)) = \widetilde{M}(D(\alpha(b))) \cong M_{\alpha(b)} \tag{3.11}$$

But ${}_BM_b$ and $M_{\alpha(b)}$ are isomorphism as B-module. This implies that ψ is an isomorphism because the homomorphism $\psi_{\mathfrak{q}}$ are al isomorphism for $\mathfrak{q} \in Y$

(4) Omitted

$\square$

Proposition 3.6
Suppose that $X = \operatorname{Spec} A$. Any sequence

$$\cdots \to M_{-1} \to M_0 \to M_1 \to \cdots \tag{3.12}$$

induces a sequence

$$\cdots \to \widetilde{M}_{-1} \to \widetilde{M}_0 \to \widetilde{M}_1 \to \cdots \tag{3.13}$$

by definition of such sheaves (this is a sequence of $\mathcal{O}_X$-modules)
Sequence **??** exact $\iff$ Sequence **??** exact

Proof
Sequence **??** exact $\iff \cdots (M_0)_p \to (M_1)_p \to \cdots$ is exact $\forall p \in X$
$\iff$ Sequence **??** is exact by looking at the stalks at p $\square$

- $\operatorname{Hom}_A(M, N) \cong \operatorname{Hom}_{\mathcal{O}_X}(\widetilde{M}, \widetilde{N}) := \{\text{morphism } \widetilde{M} \to \widetilde{N} \text{ as } \mathcal{O}_X\text{-module}\}$
(prove on 3rd Example Sheet)

Definition 3.7
X is a scheme. An $\mathcal{O}_X$-module $\mathcal{F}$ is a quasi-coherent if X can be covered by open affine schemes like $U = \operatorname{Spec} A$ s.t. $\mathcal{F}|_U = \widetilde{M}$ for some A-module M
We say that $\mathcal{F}$ is coherent if it is quasi-coherent and M is a finitely generated A-module
(M depends on U)

Example 3.8
X a scheme. Then $\mathcal{O}_X$ is a coherent sheaf:
For any open affine $U = \operatorname{Spec} A \subseteq X$, $\quad \mathcal{O}_X|_U = \mathcal{O}_U \cong \widetilde{A}$

Example 3.9
$X = \operatorname{Spec} A$ where A is a DVR. We have only 3 open sets $\emptyset, X, U = X \setminus \{\mathfrak{m}_A\} = \{\text{generic point } \eta\}$
Let $K = \kappa(\eta)$. Let $\mathcal{F}$ be the sheaf defined by $\mathcal{F}(X) = 0, \mathcal{F}(U) = K$
For any open affine covering $X = \bigcup U_i$, $\exists i$ s.t. $X = U_i$ because if $U_i \supseteq \mathfrak{m}_A \Rightarrow U_i \supseteq \eta$. There cannot be any A-module M s.t. $\mathcal{F} \cong \widetilde{M}$ because $0 = \mathcal{F}(X) = \widetilde{M}(X) = M \qquad \#$ as $M \neq 0$

Lemma 3.10
If $X = \operatorname{Spec} A$, and $\mathcal{F}$ an $\mathcal{O}_X$-module, then we have a natural morphism $\widetilde{\mathcal{F}(X)} \to \mathcal{F}$

Proof
Put $M = \mathcal{F}(X)$. We need to define a homomorphism $\widetilde{M}(U) \to \mathcal{F}(U)$ for any open set $U \subseteq X$. It is

enough to do this for U of the form $D(b), b \in A$ (as they form the base of the topology)
Now $\widetilde{M}(D(b)) \cong M_b$ by Theorem 3.3. Now define

$$\begin{aligned} \widetilde{M}(D(b)) &\to \mathcal{F}(D(b)) && (3.14) \\ \frac{m}{b^r} &\mapsto m|_{D(b)} \cdot \frac{1}{b^r} \quad (\frac{1}{b^r} \in \mathcal{O}_X(D(b))) && (3.15) \end{aligned}$$

It is easy to see that the homomorphism is well-defined.
For general U, restrict section $s \in \widetilde{M}(U)$ to open sets of the form $D(b)$, take the image of the restrictions, and finaly glue those images to get a section of $\mathcal{F}(U)$ □

Corollary 3.11
$X = \operatorname{Spec} A$, M an A-module. Then $\widetilde{M}|_{D(b)} \cong \widetilde{M_b}$

Proof
By the above lemma (**??**), there is a morphism $\widetilde{M_b} \xrightarrow{\phi} \widetilde{M}|_{D(b)}$
For any $\mathfrak{p} \in D(b)$, $\phi_\mathfrak{p} : (\widetilde{M_b})_\mathfrak{p} \cong M_\mathfrak{p} \to M_\mathfrak{p}$ is an isomorphism □

Theorem 3.12
X a scheme.

$$\begin{array}{c} \text{An } \mathcal{O}_X\text{-module } \mathcal{F} \\ \text{is quasi-coherent} \end{array} \Leftrightarrow \begin{array}{l} \forall \text{ open affine } U = \operatorname{Spec} A \subseteq X \\ \mathcal{F}|_U = \widetilde{M} \text{ for some } A\text{-module } M \end{array}$$

(If X is Noetherian, replace quasi-coherent by coherent, and require M finitely generated)

Proof
$\Leftarrow$: Trivial

$\Rightarrow$: Pick any open affine $U = \operatorname{Spec} A \subseteq X$. For any $\mathfrak{p} \in U$, $\exists$ open affine $W \subseteq X, W \ni \mathfrak{p}$ s.t. $W = \operatorname{Spec} B$, and $\mathcal{F}|_W \cong \widetilde{N}$ for some B-module N
By the corollary, we could assume that $W \subseteq U$ and it is of the form $W = D(b)$ we could also assume that $X = U$
These observations say that we can assume $X = \bigcup_{i=1}^n D(b_i)$ s.t. $\mathcal{F}|_{D(b_i)} \cong \widetilde{M_i}$ for an A_{b_i}-module M_i
Note that $D(b_i) \cap D(b_j) = D(b_i b_j)$.

$$\begin{aligned} f_i : D(b_i) &\hookrightarrow X && (3.16) \\ f_{ij} : D(b_i b_j) &\hookrightarrow X && (3.17) \end{aligned}$$

We have an exact sequence

$$0 \to \mathcal{F} \xrightarrow{\phi} \underbrace{\bigoplus_i f_{i*}\left(\mathcal{F}|_{D(b_i)}\right)}_{\mathcal{G}} \xrightarrow{\psi} \underbrace{\bigoplus_{i,j} f_{ij*}\left(\mathcal{F}|_{D(b_i b_j)}\right)}_{\mathcal{L}} \qquad (3.18)$$

- ϕ is given by: For any open $V \subseteq X, s \in \mathcal{F}(V)$, $\phi(s) = (s_i) = (s_1, \ldots, s_n)$ s.t. $s_i = s|_{D(b_i) \cap V}$
- ψ is defined by: If $(s_i) \in \mathcal{G}(V)$, $(s_i) \mapsto (s_i|_{D(b_i b_j) \cap V} - s_j|_{D(b_i b_j) \cap V})$

The sequence is exact because it is just a translation of the sheaf condition for $\mathcal{F}$
- By Theorem **??**, $f_{i*}(\mathcal{F}|_{D(b_i)}) \cong {}_A\widetilde{M_i}$, $\quad f_{ij*}(\mathcal{F}|_{D(b_j b_j)}) \cong \widetilde{M_{ij}}$ for some $A_{b_i b_j}$-module M_{ij}
- Now

$$0 \to \mathcal{F}(X) \to \bigoplus_i {}_A M_i \to \bigoplus_{i,j} M_{ij} \qquad (3.19)$$

is exact which determines an exact sequence

$$0 \to \widetilde{\mathcal{F}(X)} \to \bigoplus{}_A \widetilde{M_i} \xrightarrow{\psi} \bigoplus_{i,j} \widetilde{M_{ij}} \tag{3.20}$$

where by construction $\bigoplus{}_A \widetilde{M_i} = \mathcal{G}$ and $\bigoplus \widetilde{M_{ij}} \cong \mathcal{L}$
$\mathcal{F}$ and $\widetilde{\mathcal{F}(X)}$ are both isomorphic to the $\ker \psi$, since $\ker \psi$ is uniquely determined, then $\mathcal{F} \cong \widetilde{\mathcal{F}(X)}$

□

Theorem 3.13
X a scheme, $\phi : \mathcal{F} \to \mathcal{G}$ a morphism of quasi-coherent scheaves, then $\ker \phi$ and $\operatorname{Im} \phi$ are also quasi-coherent sheaves. If X Noetherian, then same for coherent sheaves.

Proof
If $U = \operatorname{Spec} A \subseteq X$ is open affine
$\Rightarrow$ $\exists A$-modules M, N s.t. $\mathcal{F}|_U \cong \widetilde{M}, \mathcal{G}|_U \cong \widetilde{N}$
$\Rightarrow$ $\phi|_U : \mathcal{F}|_U \to \mathcal{G}|_U$
$\Rightarrow$ have homomorphism $M \to N$

If $K = \ker(M \to N) \Rightarrow$ have exact sequence

$$0 \to \widetilde{K} \to \widetilde{M} \to \widetilde{N}$$

$\Rightarrow$ $\widetilde{K} \cong \ker \phi$ on U
$\Rightarrow$ $\ker \phi$ quasi-coherent
Similarly for $\operatorname{Im} \phi$

For Noetherian X: M, N are finite generated over A and A is Noetherian. Under the above notations, $K \leq M$ which is finitely generated over A
$\Rightarrow$ $\ker \phi$ coherent □

Remark. f is quasi-compact if open $U \subseteq Y$, then $f^{-1}U$ is quasi-compact, which means that any open cover of $f^{-1}U$ has a finite subcover

Theorem 3.14
$f : X \to Y$ a morphism of schemes. Then

(1) If $\mathcal{G}$ is quasi-coherent (resp. coherent) sheaf on Y,
then $f^* \mathcal{F}$ is quasi-coherent (resp. coherent) on X

(2) If, (Case 1) X is Noetherian
or (Case 2) f is separated and quasi-compact
then $f_* \mathcal{G}$ is quasi-coherent for any quasi-coherent $\mathcal{F}$ on X

Proof

(1) We may assume that Y is affine because the problem is local on X.
Let $Y = \operatorname{Spec} A$ and $U = \operatorname{Spec} B \subseteq X$ be an open affine set.
$\mathcal{G}$ is quasi-coherent
$\Rightarrow$ $\exists A$-module M s.t. $\mathcal{F} \cong \widetilde{M}$
$\Rightarrow$ $f^* \mathcal{G}|_U \cong \widetilde{M \otimes_A B}$ (By Theorem **??**)
$\Rightarrow$ $f^* \mathcal{G}$ is quasi-coherent.

For coherent case:
M is finitely generated over A
$\Rightarrow$ $M \otimes_A B$ is finitely generated over B hence the claim

(2) The problem is local so we may again assume that Y is affine.
Take any open affine cover of $X = \bigcup U_i$ (which we may assume to be finite)
Case 2: $U_i \cap U_j$ is again affine $\forall i, j$ (on Example Sheet 2)
Case 1: $U_i \cap U_j$ may not be affine but it can be covered by finitely many open affine sets
$$U_i \cap U_j = \bigcup U_{ijk}$$
Similar to the proof of theorem **??**, we have an exact sequence (Note that $f_i : U_i \hookrightarrow X$ and $f_{ij} : U_{ij} \hookrightarrow X$)

$$0 \to f_* \mathcal{F} \xrightarrow{\phi} \overbrace{\bigoplus f_{i*}(\mathcal{F}|_{U_i})}^{\mathcal{G}} \xrightarrow{\psi} \overbrace{\bigoplus_{ijk} f_{ijk*}(\mathcal{F}|_{U_{ijk}})}^{\mathcal{L}} \tag{3.21}$$

Now $\mathcal{G}$ and $\mathcal{L}$ are quasi-coherent on Y
$\Rightarrow$ $\ker \psi$ is quasi-coherent (by theorem **??**)
$\Rightarrow$ $f_* \mathcal{F} \cong \ker \psi$ quasi-coherent

□

Exercise: (also on Example Sheet) Find an example of morphism $f : X \to Y$ of Noetherian schemes and a coherent sheaf $\mathcal{F}$ on X s.t. $f_* \mathcal{F}$ is not coherent

Remark. Fact: If f is proper, then, $\mathcal{F}$ coherent implies $f_* \mathcal{F}$ is coherent (this is very difficult to prove, do not attempt)

Definition 3.15
X a scheme. An ideal sheaf is a $\mathcal{O}_X$-module $\mathcal{I} \subseteq \mathcal{O}_X$
If $f : Y \to X$ is a closed immersion, then the ideal sheaf of Y is $\mathcal{I}_Y = \ker(\mathcal{O}_X \to f_* \mathcal{O}_Y)$

If Z is a closed subscheme of X, then Z is given by an equivalent class of closed immersions, and the ideal sheaf of all such closed immersions are the same and so we can associate a unique ideal sheaf to Z

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \\ Z' \arrow[ur] & \end{tikzcd}$$

Theorem 3.16
X a scheme. Then, there is one-to-one correspondence between closed subschemes of X and quasi-coherent ideal sheaves

$$\{\text{Closed subschemes of } X\} \longleftrightarrow \{\text{Quasi-coherent ideal sheaves}\}$$

(If X Noetherian, same with coherent)

Proof

$\to$: If Y is a closed subscheme defined by $f : Y \to X$, then as mentioned associate the ideal sheaf which is $\ker(\mathcal{O}_X \to f_* \mathcal{O}_Y)$

U is any open affine subscheme
$\Rightarrow$ $U \cap Y$ can be covered by finite many open affine subscheme $\Rightarrow$ f quasi-compact

f closed immersion $\Rightarrow$ f separated (by Example **??**)(even proper)
$\Rightarrow$ $f_* \mathcal{O}_Y$ is quasi-coherent (by Theorem **??**)
$\Rightarrow$ $\ker(\mathcal{O}_X \to f_* \mathcal{O}_Y)$ is quasi-coherent
$\Rightarrow$ $\mathcal{I}_Y$ is quasi-coherent

$\leftarrow$: If $\mathcal{I} \subseteq \mathcal{O}_X$ is an quasi-coherent ideal sheaf then we have an exact sequence

$$0 \to \mathcal{I} \hookrightarrow \mathcal{O}_X \twoheadrightarrow \mathcal{O}_X / \mathcal{I} \to 0$$

where $\mathcal{O}_X / \mathcal{I}$ is the sheaf associated to the presheaf $U \mapsto \mathcal{O}(U)/\mathcal{I}(U)$ (where $U \subseteq X$ open)
Define $\Gamma = \{\mathfrak{p}\, X | (\mathcal{O}_X / \mathcal{I})_{\mathfrak{p}} = 0\}$

Claim: Γ open subset of X
Proof of Claim:
if $U = \operatorname{Spec} A \subseteq X$ is open affine, then $\mathcal{I}|_U = \widetilde{I}$ for some ideal $I \trianglelefteq A$
$\Gamma \cap U = U \setminus V(I)$ $\Rightarrow$ Γ is open and $Y = X \setminus \Gamma$ is closed. ■

Locally on U we have $(\mathcal{O}_X / \mathcal{I})|_U = \widetilde{A/I}$
$\Rightarrow$ can consider Y as a closed subscheme which is locally on U is given by $\operatorname{Spec} A/I \to \operatorname{Spec} A$
If $\mathcal{O}_Y$ is the locally ringed structure obtained in this way and $f : Y \hookrightarrow X$, then $\mathcal{O}_X \twoheadrightarrow f_* \mathcal{O}_Y$ because $f_* \mathcal{O}_Y = \mathcal{O}_X / \mathcal{I}$ (locally on U, $f_* \mathcal{O}_Y |_U = \widetilde{A/I}$)($\mathcal{O}_Y = f^{-1}(\mathcal{O}_X / \mathcal{I})$)
Y is the closed subscheme associated to $\mathcal{I}(= \mathcal{I}_Y)$

□

Corollary 3.17

$$\{\text{Closed subschemes of } X = \operatorname{Spec} A\} \longleftrightarrow \{\text{Ideals of } A\}$$

Proof
We use Theorem **??**: Closed subschemes of $X \leftrightarrow$ quasi-coherent ideal sheaves on X.
$\leftarrow$: $\mathcal{I} \subseteq \mathcal{O}_X$ is a quasi-coherent sheaf $\Rightarrow \begin{cases} I := \mathcal{I}(X) \subseteq \mathcal{O}_X(X) = A \\ \mathcal{I} = \widetilde{I} = \operatorname{Spec} A/I \end{cases}$
$\rightarrow$: $I \trianglelefteq A \Rightarrow \operatorname{Spec} A/I$ determines a closed subscheme of X which corresponds to the ideal sheaf $\widetilde{I}$
□

Definition 3.18
$S = \bigoplus_{d\geq 0} S_d$ a graded ring, $X = \operatorname{Proj} S$, $M = \bigoplus_{d\in\mathbb{Z}} M_d$ a graded S-module.
We associate to M a sheaf on X denoted by $\widetilde{M}$ as follows:

$$\widetilde{M}(U) = \left\{ s : U \to \bigcup_{\mathfrak{p}\in U} M_{\mathfrak{p}} \;\middle|\; \begin{array}{l} \forall\, \mathfrak{p} \in U, s(\mathfrak{p}) \in M_{\mathfrak{p}} \\ \exists \text{ open } W \subseteq U, W \ni \mathfrak{p} \\ \text{and } m \in M, a \in S \text{ homogeneous of the same degree} \\ \text{s.t. } s(\mathfrak{q}) = \frac{m}{\alpha} \in M_{\mathfrak{p}} \;\; \forall\, \mathfrak{q} \in W, \alpha \notin \mathfrak{q} \end{array} \right\}$$

$\widetilde{M}$ is an $\mathcal{O}_X$-module

Theorem 3.19
$S = \bigoplus_{d\geq 0} S_d$ a graded ring and M a graded S-module. Then,

(1) $\forall\, \mathfrak{p} \in X$, $(\widetilde{M})_{\mathfrak{p}} \cong M_{\mathfrak{p}}$

(2) $\forall b \in \bigoplus_{d>0} S_d$ homogeneous $\Rightarrow$ $\widetilde{M}|_{D_+(b)} \cong \widetilde{M_b}$ (as sheaf on affine scheme $\operatorname{Spec} S_b$)

(3) $\widetilde{M}$ is a quasi-coherent sheaf.

(4) If S is Noetherian and M finitely generated over S $\Rightarrow$ $\widetilde{M}$ is coherent.

Proof
Similar to previous theorems.
For (4), use fact from commutative algebra: M_b is finitely generated over S_b □

Definition 3.20
$S = \bigoplus_{d\geq 0} S_d$ a graded ring, $X = \operatorname{Proj} S$.
Define S-module $S(n)$ by $S(n)_d = S_{n+d}$ $(n \in \mathbb{Z})$ i.e.

$$S(n) = \bigoplus_{d\in\mathbb{Z}} S(n)_d = \bigoplus_{d\in\mathbb{Z}} S_{d+n}$$

The sheaf $\widetilde{S(n)}$ is denoted by $\mathcal{O}_X(n)$

- For $n = 0$, we get back to $S(0) = S$ and $\mathcal{O}_X(0) = \mathcal{O}_X$
- For any $\mathcal{O}_X$-module $\mathcal{F}$, set $\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$
- For M an S-module, define $M(n)$ similarly to $S(n)$

Theorem 3.21
$S = \bigoplus_{d\geq 0} S_d$ graded ring s.t. S is generated by S_1 as an algebra over S_0 (e.g. polynomial ring, or its quotients). Then,

(1) $\forall S$-modules M, N, $\quad \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \cong \widetilde{M \otimes_S N}$

(2) $\forall S$-module M, $\quad \widetilde{M}(n) := \widetilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \cong \widetilde{M(n)}$
In particular, $\quad \mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \cong \mathcal{O}_X(m+n)$

(3) Let $\alpha : S \twoheadrightarrow T$ be a surjective graded homomorphism of graded rings (i.e. preserving degrees)
Let $f : \operatorname{Proj} T \to \operatorname{Proj} S$ the corresponding morphism (see Example sheet 2)
If L is a S-module, then $f^*\widetilde{L} \cong \widetilde{L \otimes_S T}$
If K is a T-module, then $f_*\widetilde{K} \cong \widetilde{{}_S K}$

Proof

(1) We define a morphism $\phi : \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \to \widetilde{M \otimes_S N}$, since open sets of the form $D_+(b)$ ($b \in \bigoplus_{d>0} S_d$ homogeneous) give a base for the topology on X, it is enough to define ϕ on the open sets $D_+(b)$:

If open $U \subseteq X$, $\quad \gamma \in \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}(U)$
first restrict γ to open sets $D_+(b) \subseteq U$ $\qquad (\rightsquigarrow \quad \gamma|_{D_+(b)})$
then take the images of $\gamma|_{D_+(b)}$ $\qquad (\rightsquigarrow \quad \phi_{D_+(b)}(\gamma|_{D_+(b)}))$
and glue them together to determine the image of γ $\qquad (\rightsquigarrow \quad \phi_U(\gamma))$

Now pick $b \in \bigoplus_{d>0} S_d$ homogeneous.
Note that $\quad \widetilde{M}|_{D_+(b)} \cong \widetilde{M_b}, \qquad \widetilde{N}|_{D_+(b)} \cong \widetilde{N_b}$
$\Rightarrow \quad \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}|_{D_+(b)} \cong \widetilde{M}|_{D_+(b)} \otimes_{\mathcal{O}_{D_+(b)}} \widetilde{N}|_{D_+(b)} \cong \widetilde{M_b \otimes_{S_b} N_b}$
Finally, $\quad \widetilde{M \otimes_S N}|_{D_+(b)} \cong (\widetilde{M \otimes_S N})_b$

Now if $\gamma \in \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N}(D_+(b))$, then we can consider γ as an element in $M_b \otimes_{S_b} N_b$
We have a natural homomorphism

$$\begin{aligned} \theta : M_b \otimes_{S_b} N_b &\to (M \otimes_S N)_b \\ \frac{m}{b^r} \otimes \frac{n}{b^{r'}} &\mapsto \frac{m \otimes n}{b^{r+r'}} \end{aligned}$$

So we can take the $\phi_{D_+(b)}(\gamma) = \theta(\gamma)$
Since S_1 generates S as an algebra over S_0, then the open sets $D_+(b)$ with $b \in S_1$ cover X.
If $b \in S_1$, then θ is an isomorphism
$\Rightarrow \phi_{D_+(b)}$ is an isomorphism $\forall b \in S$
$\Rightarrow \phi$ is an isomorphism.

(2) Follows from (1) considering $M(n) \cong M \otimes_S S(n) \quad$ and $\quad S(m) \otimes_S S(n) = S(m+n)$

(3) L is an S-module.
Pick $c \in T_1$ and $b \in S_1$ s.t. $\alpha(b) = c$ and both of them are homogeneous of positive degree
$\Rightarrow \quad f^{-1}D_+(b) = D_+(c)$.
Define morphism $\psi : f^*\widetilde{L} \to \widetilde{L \otimes_S T}$
Similar to (1) we define it on the open sets $D_+(c)$

In this case (with slightly abuse of notation)

$$f^*\widetilde{L}|_{D_+(c)} = f^*(\widetilde{L}|_{D_+(b)}) \cong f^*(\widetilde{L_b}) \cong \widetilde{L_b \otimes_{S_b} T_c}$$

We also have $\widetilde{L \otimes_S T}|_{D_+(c)} \cong (\widetilde{L \otimes_S T})_c$
We have a homomorphism

$$\begin{array}{rcl} \mu : L_b \otimes_{S_b} T_c & \to & (L \otimes_S T)_c \\ \dfrac{l}{b^r} \otimes \dfrac{t}{c^{r'}} & \mapsto & \dfrac{l \otimes t}{c^{r+r'}} \end{array}$$

This defines ψ.
If $c \in T$, then μ is an isomorphism
$\Rightarrow \quad \psi$ is an isomorphism over $D_+(c)$
$\Rightarrow \quad \psi$ is an isomorphism (by looking at $\psi_{\mathfrak{q}}, \mathfrak{q} \in \operatorname{Proj} T$).

$\square$

Under some mild conditions, if $\mathcal{F}$ is a quasi-coherent $X = \operatorname{Proj} S$, then $\mathcal{F} = \widetilde{\Gamma_*(\mathcal{F})}$ where $\Gamma_*(\mathcal{F}) = \bigoplus_{n\in\mathbb{Z}} \mathcal{F}(n)(X)$

4 Invertible Scheaves and Cartier Divisors

(Proofs in this section will not be examinable)

Informal Discussion
X a smooth variety over algebraically closed filed k. A subvariety $D \subseteq X$ is called a <u>prime divisor</u> if $\dim D = \dim X - 1$
A <u>divisor</u> is a linear combination $\sum n_i D_i$ where $n_i \in \mathbb{Z}$ and D_i are prime divisors.
For any point $x \in X$, $\exists$ rational function f on X s.t. $D|_{U_i} = (f)|_{U_i}$ for some open set $U_i \ni x$
(End of informal discussion)

Definition 4.1
$(X, \mathcal{O}_X)$ ringed space. An $\mathcal{O}_X$-module $\mathcal{L}$ is called <u>invertible</u> if $\forall x \in X, \exists U(\ni x) \subseteq X$, s.t. $\mathcal{L}|_U \cong \mathcal{O}_U$

Theorem 4.2 (or Definition)
$(X, \mathcal{O}_X)$ ringed space. Then the set of all invertible sheaves on X (up to isomorphism) forms a group $\operatorname{Pic}(X)$ called the <u>Picard group of X</u> under the operation $\otimes$

Proof
If $\mathcal{L}$ and $\mathcal{M}$ are two invertible sheaves on $(X, \mathcal{O}_X)$, then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$ is also an invertible sheaf:
$\forall x \in X, \exists U(\ni x) \subseteq X$ s.t. $\mathcal{L}|_U \cong \mathcal{O}_U$, $\mathcal{M}|_U \cong \mathcal{O}_U$
$\Rightarrow \quad \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}|_U \cong \mathcal{O}_U \otimes_{\mathcal{O}_U} \mathcal{O}_U \cong \mathcal{O}_U \quad (A \otimes_A A \cong A)$

If $\mathcal{L}$ is invertible, put

$$\mathcal{L}^{-1} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$$

where

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)(W) = \{\mathcal{O}_X\text{-morphism } \mathcal{L}|_W \to \mathcal{O}_X|_W\}$$

(Check $\mathcal{L}^{-1}$ is invertible) We define a morphism $\phi : \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}^{-1} \to \mathcal{O}_X$ by taking homomorphisms

$$\begin{array}{rcl} \phi : \mathcal{L}(W) \otimes_{\mathcal{O}_X(W)} \mathcal{L}^{-1}(W) & \to & \mathcal{O}_X(W) \\ l \otimes \theta & \mapsto & \theta_W(l) \end{array}$$

where

$$\begin{array}{rcl} \theta : \mathcal{L}|_W & \to & \mathcal{O}_X|_W \\ \theta_W : \mathcal{L}|_W(W) & \to & \mathcal{O}_X|_W(W) \end{array}$$

If $\mathcal{L}|_U \cong \mathcal{O}_U$ and $\mathcal{L}^{-1}|_U \cong \mathcal{O}_U$
$\Rightarrow$ ϕ_U looks like isomorphism$\mathcal{O}_U \otimes_{\mathcal{O}_U} \mathcal{O}_U \xrightarrow{\sim} \mathcal{O}_U$
$\Rightarrow$ ϕ is an isomorphism locally
$\Rightarrow$ ϕ is an isomorphism
(Note that by this construction, $\mathcal{O}_X$ is the identity element in $\mathrm{Pic}(X)$) $\square$

Exercises:

(1) Work out the theorem and proof assuming that X affine

(2) $X = \mathrm{Proj}\, S$ for a graded ring $S = \bigoplus_{d \geq 0} S_d$ which is generated by S_1 over S_0. Then $\mathcal{O}_X(n)$ are invertible sheaves

Definition 4.3
X is an integral. K=function field of X=$(\mathcal{O}_X)_\eta$ where η is the generic point (0). For any open $U \subseteq X$, $\mathcal{O}_X(U) \hookrightarrow K$

- A Cartier divisor D is a system (U_i, f_i) where U_i cover X and $f_i \in K$ s.t.
  $\forall i, j \quad f_i/f_j \in \mathcal{O}_X^*(U_i \cap U_j) = \{\text{invertible elements of } \mathcal{O}_X(U_i \cap U_j)\}$

Here two systems (U_i, f_i) and (V_α, g_α) define the same Cartier divisor if $\forall i, \alpha \quad f_i/g_\alpha \in \mathcal{O}_X^*(U_i \cap V_\alpha)$

If D and D' are Cartier divisors given by (U_i, f_i) and (U'_α, f'_α) respectively
then $D + D'$ is defined by the system $(U_i \cap U'_\alpha, f_i f'_\alpha)$ (we could assume we have the same open sets)
(this product is not composition)

We also defines $-D$ to be given by (U_i, f_i^{-1})
We say D principal if it is given by (X, f)

We define the Cartier divisor group

$$\mathrm{Div}(X) = \frac{\text{group of all Cartier divisors under } +}{\text{Principle Cartier divisors}}$$

Definition 4.4
X integral scheme. K=function field of X. D a Cartier divisor on X, given by (U_i, f_i).
Then associate to D a sheaf $\mathcal{O}_X(D)$ as follows:

$$\forall \text{ open } U \subseteq X, \quad \mathcal{O}_X(D)(U) = \{h \in K | h f_i \in \mathcal{O}_X(U_i \cap U) \ \forall i\}$$

Note that if D is also given by (V_α, g_α)
$\Rightarrow$ if $h \in \mathcal{O}_X(D)(U)$, then $h f_i \in \mathcal{O}_X(U_i \cap U)$
$\Rightarrow$ $h g_\alpha|_{U \cap U_i \cap V_\alpha} = (h f_i)(f_i g_\alpha)|_{U \cap U_i \cap V_\alpha}$ where $h f_i|_{U \cap U_i \cap V_\alpha} \in \mathcal{O}_X(U \cap U_i \cap V_\alpha)$
and $f_i^{-1} g_\alpha|_{U \cap U_i \cap V_\alpha} \in \mathcal{O}_X^*(U \cap U_i \cap V_\alpha)$

$\Rightarrow \quad hg_\alpha|_{U\cap U_i\cap V_\alpha} \in \mathcal{O}_X(U\cap U_i\cap V_\alpha)$
Also notice $U\cap U_i\cap V_\alpha$ gives a covering of $U\cap V_\alpha$
$\Rightarrow \quad hg_\alpha \in \mathcal{O}_X(U\cap V_\alpha)$
$\therefore \quad$ if $h\in\mathcal{O}_X(D)(U)$ wrt (U_i,f_i) $\Rightarrow$ $h\in\mathcal{O}_X(D)(U)$ wrt (V_α,g_α)
(the other direction is similar)

Theorem 4.5
X integral scheme. Then,

(1) If D Cartier divisor, then $\mathcal{O}_X(D)$ is an invertible sheaf

(2) If D, D' are Cartier divisors then $\mathcal{O}_X(D+D')\cong\mathcal{O}_X(D)\otimes_{\mathcal{O}_X}\mathcal{O}_X(D')$

(3) If D Cartier divisor, then $\mathcal{O}_X(-D)=\mathcal{O}_X(D)^{-1}$

(4) $D-D'$ principal $\quad\Leftrightarrow\quad \mathcal{O}_X(D)\cong\mathcal{O}_X(D')$

Proof

(1) If D is given by (U_i,f_i), then $\mathcal{O}_X(D)|_{U_i}=\mathcal{O}_{U_i}\cdot\frac{1}{f_i}$
where $\mathcal{O}_{U_i}\cdot\frac{1}{f_i}$ is considered as a module over $\mathcal{O}_{U_i}$ with

$$\mathcal{O}_{U_i}\cdot\frac{1}{f_i}(V)=\mathcal{O}_X(V)\cdot\frac{1}{f_i}$$

$(s=hf_i\in\mathcal{O}_X(V) \quad h=s/f_i=s\cdot\frac{1}{f_i})$
Thus, $\mathcal{O}_X(D)$ is invertible

(2) Define a morphism $\psi:\mathcal{O}_X(D)\otimes_{\mathcal{O}_X}\mathcal{O}_X(D')\to\mathcal{O}_X(D+D')$ where for any open $V\subseteq X$, we define

$$\begin{array}{rcl}\mathcal{O}_X(D)(V)\otimes_{\mathcal{O}_X(V)}\mathcal{O}_X(D')(V) & \to & \mathcal{O}_X(D+D')(V)\\ h\otimes h' & \mapsto & hh'\end{array}$$

We may assume D and D' are given by (U_i,f_i) and (U_i,f_i')
$\Rightarrow \quad D+D'$ is given by (U_i,f_i+f_i')
$\Rightarrow \quad hf_i\in\mathcal{O}_X(V\cap U_i) \quad$ and $\quad h'f_i'\in\mathcal{O}_X(U_i\cap V) \quad \forall i$
$\Rightarrow \quad hh'f_if_i'\in\mathcal{O}_X(U_i\cap V)$
$\Rightarrow \quad hh'\in\mathcal{O}_X(D+D')(V)$

(3) Exercise/Ignore.

(4) Exercise/Ignore.

$\square$

5 Differential Forms

(Proofs in this section is not examinable)

Definition 5.1
B an A-algebra (Convention: $\alpha : A \to B$, and a is identified with $\alpha(a)$)
Define <u>the module of differentials of B over A</u> as follows:
For each $b \in B$, take a symbol db
Take M to be the free B-module operated by all the db (something like $\bigoplus_B B$)
We let $\Omega_{B/A} = M/N$ where N is the submodule generated by

$$\begin{cases} da & a \in A \\ d(b+b') - db - db' & b, b' \in B \\ d(bb') - bdb' - b'db & b, b' \in B \end{cases} \tag{5.1}$$

- We have a map

$$\begin{aligned} d : B &\to \Omega_{B/A} && (5.2) \\ b &\mapsto db && (5.3) \end{aligned}$$

- $\Omega_{B/A}$ satisfies the following universal property:
  if $\theta : B \to K$ is a map and K is B-module s.t.
  $\theta(a) = 0\ \forall a \in A \qquad \theta(b+b') = \theta(b) + \theta(b') \qquad \theta(bb') = b\theta(b') + b'\theta(b)$
  Then $\exists! \mu : \Omega_{B/A} \to K$ s.t.

$$\begin{tikzcd} B \arrow[r, "d"] \arrow[d, "\theta"'] & \Omega_{B/A} \arrow[dl, dashed, "\exists!\mu"] \\ K & \end{tikzcd}$$

  diagram commutes

Example 5.2
$B = A[t_1, \ldots, t_n]$. Then

$$\Omega_{B/A} = \bigoplus_1^n B \tag{5.4}$$

Every element in $\Omega_{B/A}$ can be written as a sum

$$\sum f_i dt_i \qquad f_i \in B \tag{5.5}$$

(It turns out that the f_i are determined uniquely)

Definition 5.3
$f : \operatorname{Spec} B = X \to Y = \operatorname{Spec} A$ a morphism of *affine* schemes

(1) Define the <u>sheaf of relative differential forms</u>, $\Omega_{X/Y} := \widetilde{\Omega_{B/A}}$ (sheaf associated to the module)

(2) For general schemes (not necessarily affine) X, Y, then define $\Omega_{X/Y}$ as follows:
Cover Y with open affine scheme U_i.
Cover $f^{-1}U_i$ with open affine schemes $V_{i,\alpha}$, so we have morphism $V_{i,\alpha} \to U_i \quad \forall i, \alpha$
For each i and α take the sheaf $\Omega_{V_{i,\alpha}/U_i}$. It turns out that one can glue all these sheaves to obtain sheaf on X, i.e. $\Omega_{X/Y}$

Example 5.4
$X = \mathbb{A}^n_A$, $Y = \operatorname{Spec} A$, $B = A[t_1, \ldots t_n]$. Then

$$\Omega_{X/Y} = \widetilde{\Omega_{B/A}} = \bigoplus_1^n \widetilde{B} \tag{5.6}$$

Theorem 5.5
$\begin{array}{l} f : X \to Y \\ g : S \to Y \end{array}$ morphisms
$$\begin{tikzcd} S \times_Y X \arrow[r, "h"] \arrow[d] & X \arrow[d, "f"] \\ S \arrow[r, "g"] & Y \end{tikzcd}$$
Then

$$\Omega_{S \times_Y X/S} = h^* \Omega_{X/Y} \tag{5.7}$$

Theorem 5.6
$\begin{array}{l} f : X \to Y \\ g : Y \to Z \end{array}$ morphisms. Then there is an exact sequence

$$f^* \Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0 \tag{5.8}$$

Theorem 5.7
A a ring, $X = \mathbb{P}^n_A = \operatorname{Proj} A[t_0, \ldots, t_n]$, $Y = \operatorname{Spec} A$. Then there is an exact sequence

$$0 \to \Omega_{X/Y} \to \bigoplus_1^{n+1} \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0 \tag{5.9}$$

Proof
Take $S = A[t_0, \ldots, t_n]$. Define an S-module to $\bigoplus_1^{n+1} S(-1)$

$$\begin{aligned} S : \quad & S_0 \oplus S_1 \oplus S_2 \oplus \cdots && (5.10) \\ S(-1) : \quad & 0 \oplus S_0 \oplus S_1 \oplus \cdots && (5.11) \\ S^{n+1}(-1) : \quad & 0 \oplus S_0^{n+1} \oplus S_1^{n+1} \oplus \cdots && (5.12) \\ & && (5.13) \end{aligned}$$

$$S_0^{n+1} = \bigoplus_1^{n+1} A \tag{5.14}$$

$$e_0 = (1, 0, \ldots, 0) \qquad e_1 = (0, 1, 0, \ldots, 0) \tag{5.15}$$

Also define an S-homomorphism

$$\begin{aligned} \alpha : T &\to S && (5.16) \\ e_i &\mapsto t_i && (5.17) \end{aligned}$$

Let $M = \ker \alpha$, then we have an exact sequence

$$0 \to M \to T \xrightarrow{\alpha} S \tag{5.18}$$

and thus

$$0 \to \widetilde{M} \to \widetilde{T} \to \widetilde{S} \to 0 \tag{5.19}$$

It turns out that $\Omega_{X/Y} \cong \widetilde{M}$

Define morphisms $\phi_i : \Omega_{X/Y}|_{U_i} \to \widetilde{M}|_{U_i}$ (where $U_i = D_+(t_i)$)
Know that $\Omega_{X/Y}|_{U_i} = \Omega_{U_i/Y} = \widetilde{\Omega_{B_i/A}}$ $\quad (B_i = S_{t_i} = A[t_0/t_i, \ldots, t_n/t_i])$
and $\Omega_{B_i/A}$ is generated by $d(t_k/t_i)$

Define a homomorphism

$$
\begin{aligned}
h_i : \Omega_{B_i/A} &\rightarrow M_{t_i} && (5.20)\\
d\,(t_k/t_i) &\mapsto \frac{1}{t_i^2}(t_i e_k - t_k e_i) \in M && (5.21)
\end{aligned}
$$

It turns out that h_i is an isomorphism. If ϕ_i defined by h_i, then ϕ_i is also an isomorphism

We have $\frac{t_k}{t_i} = \frac{t_k}{t_j}\frac{t_j}{t_i}$ on $U_i \cap U_j$ which implies

$$
d\left(\frac{t_k}{t_i}\right) = \frac{t_k}{t_j} d\left(\frac{t_j}{t_i}\right) + \frac{t_j}{t_i} d\left(\frac{t_k}{t_j}\right) \tag{5.22}
$$

Therefore we can glue the ϕ_i to obtain morphism $\phi : \Omega_{X/Y} \rightarrow \widetilde{M}$
(Remember that in general, we cannot glue morphisms) $\square$

Example 5.8
X a variety over an algebraically closed field K: an integral, separated scheme of finite type over $\operatorname{Spec} K$
Then X is smooth (i.e. $\mathcal{O}_x$ are regular rings $\forall x \in X$)
Then $\Omega_{X/\operatorname{Spec} K}$ is locally free i.e. locally it is isomorphic to $\bigoplus \mathcal{O}$
The converse also holds: If $\Omega_{X/\operatorname{Spec} K}$ is locally free then X is smooth

- X smooth variety over K. We can define $\mathcal{T}_X = \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/K}, \mathcal{O}_X)$
  Canonical sheaf $\omega_x = \bigwedge^{\dim X} \Omega_{X/K}$

6 Cohomology

Definition 6.1
An Abelian Category is a category which behave like the category of abelian groups, for examples:

- The category, $\mathcal{A}b$, of abelian groups
- The category, A-Mod, of modules over a ring A
- The category, $\mathcal{S}h(X)$, of sheaves of abelian groups on a topological space X
- The category, $\mathcal{M}(X)$, of $\mathcal{O}_X$-modules on a ringed space $(X, \mathcal{O}_X)$
- The category of quasi-coherent sheaves on a scheme X

Definition 6.2
$\mathcal{A}$ an abelian category. A complex in $\mathcal{A}$ is a sequence

$$A^\bullet : \qquad \cdots \to A^i \xrightarrow{d^i} A^{i+1} \to \cdots \tag{6.1}$$

s.t. $\operatorname{Im} d^{i-1} \subseteq \ker d^i$, with A^i objects in $\mathcal{A}$ and d^i morphisms
The i-th cohomology of $A^\bullet$ is

$$h^i(A^\bullet) = \frac{\ker d^i}{\operatorname{Im} d^{i-1}} \tag{6.2}$$

- A morphism $f : A^\bullet \to B^\bullet$ is a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A^i \arrow[r] \arrow[d, "f^i"'] & A^{i+1} \arrow[r] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r] & B^i \arrow[r] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

- A short exact sequence $\bullet \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ is a sequence of complexes s.t. the induced sequence $0 \to A^i \to B^i \to C^i \to 0$ are exact $\forall i$

- For each short exact sequence, we get a long exact sequence of cohomology:

$$\cdots \to h^i(A^\bullet) \to h^i(B^\bullet) \to h^i(C^\bullet) \to h^{i+1}(A^\bullet) \to h^{i+1}(B^\bullet) \to \cdots \tag{6.3}$$

  (c.f. Algebraic Topology, Snake Lemma. Note the cohomology in Algebraic Topology is in reverse direction)

Definition 6.3
$F : \mathcal{A} \to \mathcal{B}$ a covariant functor between abelian categories

- F is called additive if the map induced $\operatorname{Hom}(A, A') \to \operatorname{Hom}(F(A), F(A'))$ is a homomorphism of abelian groups

- F is left exact if

  (1) F is additive
  (2) if $\quad 0 \to A' \to A \to A'' \to 0 \quad$ is a s.e.s in $\mathcal{A}$
  then $\quad 0 \to F(A') \to F(A) \to F(A'') \quad$ is exact

Definition 6.4
$\mathcal{A}$ an abelian category and I an object in $\mathcal{A}$
I is injective if for any diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d] & A' \arrow[dl, dashed, "\exists g"] \\ & I & \end{tikzcd}$$

with the top row sequence being exact, then $\exists g : A' \to I$ which fits into the diagram.

A an object in $\mathcal{A}$. An injective resolution for A is an exact sequence $0 \to A \to I^0 \to I^1 \to \cdots$ s.t. I^j are injective

Theorem 6.5 (and Definition)
$F : \mathcal{A} \to \mathcal{B}$ a left exact functor between abelian categories.
Assume that $\mathcal{A}$ has enough injectives, i.e. every object in $\mathcal{A}$ has an injective resolution.

- Pick A an object in $\mathcal{A}$ and pick an injective resolution: $0 \to A \to I^0 \to I^1 \to \cdots$
  Apply F to get the complex: $D^\bullet : \quad 0 \to F(I^0) \to F(I^1) \to \cdots$
  We define the right derived functors of F to be

$$\begin{aligned} R^iF : \mathcal{A} &\to \mathcal{B} && (6.4) \\ A &\mapsto h^i(D^\bullet) && (6.5) \end{aligned}$$

  (For this course, think of $\mathcal{A} = \mathcal{S}h(X)$, $\mathcal{B} = \mathcal{A}b$)

Under the above assumptions and notations, we have:

(1) $R^iF(A)$ does not depend on the choice of injective resolution for A

(2) $R^0F \cong F$ (trivial)

(3) Any s.e.s $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$ induces a (canonical) long exact sequence of cohomology

$$0 \to R^0F(A') \to \cdots \to R^iF(A') \to R^iF(A) \to R^iF(A'') \to R^{i+1}F(A') \to \cdots \tag{6.6}$$

(4) Any commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C' \arrow[r] & C \arrow[r] & C'' \arrow[r] & 0 \end{tikzcd}$$

(top and bottom rows are s.e.s), then it induces a commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & R^iF(A') \arrow[r] \arrow[d] & R^iF(A) \arrow[r] \arrow[d] & R^iF(A'') \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & R^iF(C') \arrow[r] & R^iF(C) \arrow[r] & R^iF(C'') \arrow[r] & \cdots \end{tikzcd}$$

(5) An object J in $\mathcal{A}$ is called F-acyclic if $R^iF(J) = 0 \; \forall i > 0$.
If A an object in $\mathcal{A}$ and

$$0 \to A \to J^0 \to J^1 \to \cdots \tag{6.7}$$

exact sequence and J^i F-acyclic.
Then $R^iF(A) \cong h^i(E^\bullet)$ where

$$E^\bullet : \qquad 0 \to F(J^0) \to F(J^1) \to \cdots \tag{6.8}$$

Remark. If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is a s.e.s
then $0 \to \mathcal{F}'(X) \to \mathcal{F}(X) \to \mathcal{F}''(X)$ is exact (i.e. global section is a left exact functor)

Definition 6.6
X a topological space and $\mathcal{S}h(X)$ the category of sheaves on X
Let $\mathcal{A}b$ be the category of abelian groups and functor

$$\begin{aligned} F : \mathcal{S}h(X) &\to \mathcal{A}b && (6.9) \\ \mathcal{F} &\mapsto \mathcal{F}(X) && (6.10) \end{aligned}$$

(Note the remark above)
We define the <u>i-th cohomology group of $\mathcal{F}$</u>

$$H^i(X, \mathcal{F}) = R^i F(\mathcal{F}) \tag{6.11}$$

This definition is justified by the following theorem

Theorem 6.7
$\mathcal{S}h(X)$ has enough injectives

Proof
First, note that we can consider X as a ringed space $(X, \mathcal{O}_X)$ where $\mathcal{O}_X$ = constant sheaf on X defined by $\mathbb{Z}$ (c.f. Example Sheet 3)
We could instead try to proof the theorem for $\mathcal{M}(X)$

Pick $\mathcal{F} \in \mathcal{M}(X)$. The stalk $\mathcal{F}_x$ is a module over $\mathcal{O}_X$
By commutative algebra, there is an exact sequence

$$0 \to \mathcal{F}_x \to I_x \tag{6.12}$$

where I_x is an injective module over $\mathcal{O}_X$

I_x can be considered as a sheaf on $\{x\}$
If $f_x : \{x\} \hookrightarrow X$ is the inclusion map $\Rightarrow$ $f_{x*} I_x$ is an $\mathcal{O}_X$-module.
Now let

$$\mathcal{I} = \prod_{x \in X} f_{x*} I_x \tag{6.13}$$

It turns out that $\mathcal{I}$ is injective in $\mathcal{M}(X)$
Note that for any $\mathcal{G} \in \mathcal{M}(X)$, we have

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}) = \prod \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, f_{x*} I_x) = \prod \mathcal{H}om_{\mathcal{O}_x}(\mathcal{G}_x, I_x) \tag{6.14}$$

where the first equality is by universal property to product,
the last equality is by following:
$\phi : \mathcal{G} \to f_{x*} I_x$ is a morphism of $\mathcal{O}_X$-modules $\Rightarrow$ have $\phi_x : \mathcal{G}_x \to I_x \in \mathrm{Hom}_{\mathcal{O}_x}(\mathcal{G}_x, I_x)$

Conversely, if we are given an $\mathcal{O}_x$-homomorphism $\mathcal{G}_x \to I_x$
Then the homomorphism composed by $\mathcal{G}(U) \to \mathcal{G}_x \to = (f_{x*} I_x)(U)$ if $x \in U \subseteq X$
(if $x \notin U$, then $\mathcal{G}(U) \to (f_{x*} I_x)(U) = 0$)
So we get a morphism $\mathcal{G} \to f_{x*} I_x$
In particular, by construction, $\forall x \in X$, $\exists$ hom $\mathcal{F}_x \hookrightarrow I_x$
$\Rightarrow$ by above arguments, get a injective morphism $\mathcal{F} \hookrightarrow \mathcal{I}$

$\mathcal{I}$ is injective in $\mathcal{M}(X)$:
Suppose that we have a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[d, "\phi"] & \mathcal{H} \\ & \mathcal{I} & \end{tikzcd}$$

with top row exact, and

$$
\begin{array}{ccccc}
\phi \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G},\mathcal{I}) & \overset{\text{correspond}}{\longleftrightarrow} & \prod \phi^x & & \\
 & & \phi^x \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, f_{x*}I_x) & \overset{\text{correspond}}{\longleftrightarrow} & \phi_x : \mathcal{G}_x \to \mathcal{I}_x
\end{array}
\tag{6.15}
$$

So we have

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{G}_x \arrow[r] \arrow[d, "\phi_x"'] & \mathcal{H}_x \arrow[dl, dashed, "\exists\theta"] \\
 & I_x &
\end{tikzcd}
$$

(top row exact)
I_x is injective over $\mathcal{G}_x \Rightarrow \exists\, \theta : \mathcal{H}_x \to I_x$ fitting the diagram

The above arguments say that all such homomorphism fitting the diagram determine a morphism $\mathcal{H} \to \mathcal{I}$ making the following commutative:

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{G} \arrow[r] \arrow[d, "\phi"'] & \mathcal{H} \arrow[dl] \\
 & \mathcal{I} &
\end{tikzcd}
$$

Finally, we can construct an injection resolution for $\mathcal{F}$ as follows:

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{F} \arrow[r, hook] \arrow[d] & \mathcal{I}^0 \arrow[r] & \mathcal{I}^1 \arrow[r] \arrow[d] & \cdots \\
 & \mathcal{I}^0 / \mathcal{F} \arrow[r] \arrow[ur, hook] & 0 & \mathcal{I}^1 / (\mathcal{I}^0 / \mathcal{F}) \arrow[ur, hook] &
\end{tikzcd}
$$

□

Summary:
X topological space, $\mathcal{S}h(X)$ the category of sheaves on X
$\forall\ \mathcal{F} \in \mathcal{S}h(X)$, take an injection resolution $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$
to get a complex $0 \to \mathcal{F}(X) \to \mathcal{I}^0(X) \xrightarrow{d^0} \mathcal{I}^1(X) \xrightarrow{d^1} \cdots$

$$
H^i(X,\mathcal{F}) = \begin{cases} \dfrac{\ker d^i}{\operatorname{Im} d^{i-1}} & i > 0 \\ \mathcal{F}(X) & i = 0 \end{cases}
\tag{6.16}
$$

Definition 6.8
X topological space. $\mathcal{F} \in \mathcal{S}h(X)$ is called <u>flasque</u> if for any open $U \subseteq X$, have surjection $\mathcal{F}(X) \twoheadrightarrow \mathcal{F}(U)$

Theorem 6.9
$(X, \mathcal{O}_X)$ a ringed space. Then any injection $\mathcal{I} \in \mathcal{M}(X)$ is flasque

Proof
Let $U \subseteq X$ be any open subset. Define a sheaf $\mathcal{F} \in \mathcal{M}(X)$ by

$$
\mathcal{F}(W) = \begin{cases} 0 & \text{if } W \nsubseteq U \\ \mathcal{O}_X(W) & \text{if } W \subseteq U \end{cases}
\tag{6.17}
$$

Pick $t \in \mathcal{I}(U)$, then we have morphism $\mathcal{F} \to \mathcal{I}$ by

$$\begin{cases} \mathcal{F}(W) \xrightarrow{\text{zero map}} \mathcal{I}(W) & \text{if } W \nsubseteq U \\ \begin{array}{ccc} \mathcal{F}(W) & \longrightarrow & \mathcal{I}(W) \\ 1 & \longmapsto & t|_W \end{array} & \text{if } W \subseteq U \end{cases} \tag{6.18}$$

Now we have a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r, hook] \arrow[d, "\phi_t"'] & \mathcal{O}_X \arrow[dl, dashed, "\exists\psi_t"] \\ & \mathcal{I} & \end{tikzcd}$$

and since $\mathcal{I}$ is injective, $\exists\ \psi_t : \mathcal{O}_X \to \mathcal{I}$ fitting into the diagram
Let $s \in \mathcal{I}(X) = (\psi_t)_X(1)$. Then we have a diagram

$$\begin{tikzcd} & & & 0 \arrow[r] & \mathcal{F}(X) \arrow[r, "1 \mapsto 1"] \arrow[dlll] \arrow[d] & \mathcal{O}_X(X) \arrow[dlll] \arrow[dl, "1 \mapsto s"] \\ 0 \arrow[r] & \mathcal{F}(U) \arrow[r] \arrow[d, "1 \mapsto t"] & \mathcal{O}_X(U) & & \mathcal{I}(X) \arrow[dlll, "s|_U = t"] & \\ & \mathcal{I}(U) & & & & \end{tikzcd}$$

□

Theorem 6.10
X a topological space, and $\mathcal{F} \in \mathcal{S}h(X)$ flasque.
Then $H^i(X, \mathcal{F}) = 0 \quad \forall\, i > 0$

Proof
We have an exact sequence $\quad 0 \to \mathcal{F} \hookrightarrow \mathcal{I} \to \mathcal{G} = \mathcal{I}\,/\,\mathcal{F} \to 0$
which gives an exact sequence $\quad 0 \to \mathcal{F}(X) \to \mathcal{I}(X) \xrightarrow{\alpha} \mathcal{G}(X)$

Claim: α is surjective

Proof of Claim:
Pick $t \in \mathcal{G}(X)$
$\mathcal{I} \twoheadrightarrow \mathcal{G}$ surjective $\Rightarrow$ surjectivity on sections locally

Suppose open $U_1, U_2 \subseteq X$ and $s_i \in \mathcal{I}(U_i)$ s.t. $s_i \mapsto t|_{U_i}$
$\Rightarrow \quad s_1|_{U_1 \cap U_2} - s_2|_{U_1 \cap U_2} \in \ker(\mathcal{I}(U_1 \cap U_2) \to \mathcal{G}(U_1 \cap U_2))$

On the other hand, $\ker(\mathcal{I}(U_1 \cap U_2) \to \mathcal{G}(U_1 \cap U_2) = \mathcal{F}(U_1 \cap U_2))$

$$\mathcal{F} \text{ is flasque } \Rightarrow \begin{array}{l} \mathcal{F}(X) \twoheadrightarrow \mathcal{F}(U_1 \cup U_2) \\ \mathcal{F}(X) \twoheadrightarrow \mathcal{F}(U_1 \cap U_2) \end{array} \Rightarrow \mathcal{F}(U_1 \cup U_2) \twoheadrightarrow \mathcal{F}(U_1 \cap U_2)$$
$$\Rightarrow \quad \exists r \in \mathcal{F}(U_1 \cup U_2) \quad \text{s.t.} \quad r|_{U_1 \cap U_2} = s_1|_{U_1 \cap U_2} - s_2|_{U_1 \cap U_2}$$

Note that $s_2 + r|_{U_2} \in \mathcal{I}(U_2)$ and $s_1 \in \mathcal{I}(U_1)$ restrict to the same thing in $\mathcal{I}(U_1 \cap U_2)$:

$$s_2 + r|_{U_1 \cap U_2} = s_2|_{U_1 \cap U_2} + s_1|_{U_1 \cap U_2} - s_2|_{U_1 \cap U_2} = s_1|_{U_1 \cap U_2} \tag{6.19}$$

$$\therefore \quad \exists s \in \mathcal{I}(U_1 \cup U_2) \quad \text{s.t.} \quad s \mapsto t|_{U_1 \cup U_2} \qquad \blacksquare$$

Now look at the set of pairs (U, s) s.t. $s (\in \mathcal{I}(U)) \mapsto t|_U$

Define an order: $(U_1, s_1) \leq (U_2, s_2)$ if $\begin{cases} U_1 \subseteq U_2 \\ s_2|_{U_1} = s_1 \end{cases}$

Zorn' Lemma $\Rightarrow \exists$ maximial element (U, s)

Claim: $U = X$

Proof of Claim:
Otherwise, $\exists x \in X \setminus U$ and some neighbourhood V of x and $s' \in \mathcal{I}(V)$ s.t. (V, s') is a pair.

But the above arguments show that $\exists s'' \in \mathcal{I}(U \cup V)$ s.t. $((U \cup V), s'')$ is a pair
and $(U, s) \lneqq (U \cup V, s'')$ # maximality of (U, s) ✓(End of subclaim)

$$\therefore \quad \exists s \in \mathcal{F}(X) \quad \text{s.t. } s \mapsto t \in \mathcal{G}(X) \qquad \text{(End of claim)}\blacksquare$$

The same argument shows that $0 \to \mathcal{F}(W) \to \mathcal{I}(W) \to \mathcal{G}(W) \to 0$ is exact $\forall$ open $W \subseteq X$

Claim: $\mathcal{G}$ is flasque

Proof of Claim:
If $U \subseteq X$ is open then

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(X) \arrow[r] \arrow[d, "a"] & \mathcal{I}(X) \arrow[r] \arrow[d, "b"] & \mathcal{G}(X) \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & \mathcal{F}(U) \arrow[r] & \mathcal{I}(U) \arrow[r] & \mathcal{G}(U) \arrow[r] & 0
\end{tikzcd}$$

c is surjective because a and b surjective ($\mathcal{I}$ injection $\Rightarrow$ flasque by Theorem **??**) $\blacksquare$

Finally, we have a long exact sequence:

$$\begin{aligned}
0 \longrightarrow H^0(X, \mathcal{F}) &\longrightarrow H^0(X, \mathcal{I}) \longrightarrow H^0(X, \mathcal{G}) \\
\longrightarrow H^1(X, \mathcal{F}) &\longrightarrow H^1(X, \mathcal{I}) \longrightarrow H^1(X, \mathcal{G}) \\
\longrightarrow H^2(X, \mathcal{F}) &\longrightarrow \cdots
\end{aligned}$$

$\mathcal{I}$ is injective $\Rightarrow$ acyclic $\Rightarrow H^i(X, \mathcal{I}) = 0 \ \forall i > 0$

$0 \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{I}) \to H^0(X, \mathcal{G}) \to 0$ is exact $\Rightarrow H^1(X, \mathcal{F}) = 0$
(similarly $H^1(X, \mathcal{G}) = 0$)
$\Rightarrow \quad H^i(X, \mathcal{F}) = 0 \ \forall i > 0$ by simply looking at the long exact sequence □

Corollary 6.11
X topological space, $\mathcal{F}$ a sheaf on X.
Then, cohomology of $\mathcal{F}$ can be computed using "flasque resolutions":
If we have an exact sequence $0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots$ where $\mathcal{I}^i$ are flasque, then

$$H^i(X, \mathcal{F}) = \frac{\ker d^i}{\operatorname{Im} d^{i-1}} \tag{6.20}$$

where $0 \to \mathcal{I}^0(X) \xrightarrow{d^0} \mathcal{I}^1(X) \to \cdots$

Proof
Follows immediately from Theorem **??** and Definition/Theorem **??** □

Corollary 6.12
$(X, \mathcal{O}_X)$ a ringed space, $\mathcal{F} \in \mathcal{M}(X)$. Here we have two different categories $\mathcal{M}(X)$ and $\mathcal{S}h(X)$. Then, cohomology of $\mathcal{F}$ computed with respect to $\mathcal{M}(X)$ and $\mathcal{S}h(X)$ are the same.

Proof
To compute the cohomology of $\mathcal{F}$ with respect to $\mathcal{M}(X)$, we do this:
We take an injective resolution

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots \tag{6.21}$$

and get $0 \to \mathcal{I}^0(X) \to \mathcal{I}^1(X) \to \cdots$
whose cohomology groups are defined to be the cohomology groups of $\mathcal{F}$
Theorem **??** $\Rightarrow$ $\mathcal{I}^i$ are flasque
Corollary **??** $\Rightarrow$ injective resolution (**??**) gives the cohomology groups $H^i(X, \mathcal{F})$ □

Definition 6.13
X a topological space. We say that the <u>dimension of X</u>, denoted $\dim X$, is the number s.t. if $X_0 \subsetneq \cdots \subsetneq X_n \subseteq X$ is a sequence of irreducible closed subsets of X, then $n \leq \dim X + 1$

Theorem 6.14
X a topological space of dimension d. Then

(1) $H^i(X, \mathcal{F}) = 0$ if $i > d$
(2) $\mathcal{F} \in \mathcal{S}h(X)$

(Proof omitted)

Lemma 6.15
A a Noetherian ring, I an injective A-module. Then $\widetilde{I}$ is a flasque sheaf

(Proof omitted)

Theorem 6.16
X a Noetherian scheme. Then, the following are equivalent:

(1) X is affine
(2) $H^i(X, \mathcal{F}) = 0$ if $i > 0$, $\mathcal{F}$ a quasi-coherent sheaf
(3) $H^i(X, \mathcal{F}) = 0$ if $i > 0$, $\mathcal{F}$ a coherent ideal sheaf

Proof
<u>(1) $\Rightarrow$ (2)</u>:
X is affine and Noetherian $\Rightarrow$ $X = \operatorname{Spec} A$ where A is Noetherian.
Moreover, $\mathcal{F} = \widetilde{M}$ where M is an A-module. We can take an injective resolution

$$0 \to M \to I^0 \to I^1 \to \cdots \tag{6.22}$$

which gives an exact sequence

$$0 \to \widetilde{M}(= \mathcal{F}) \to \widetilde{I^0} \to \widetilde{I^1} \to \cdots \tag{6.23}$$

which is a flasque resolution of $\mathcal{F}$ by Lemma **??**.
Since the sequence (**??**) is exact
$\Rightarrow$ $H^i(X, \mathcal{F}) = 0$ if $i > 0$ by Corollary **??**

<u>(2) $\Rightarrow$ (3)</u>: Trivial

<u>(3) $\Rightarrow$ (1)</u>:
First, for $b \in \mathcal{O}_X(X)$, we define $D(b) = \{x \in X | b \in \mathcal{O}_x \setminus \mathfrak{m}_x\}$
Note that if X is affine, this is the same as the $D(b)$ we defined before

We use the result of Question 2 on Example Sheet 3:

$$X \text{ affine} \Leftrightarrow \begin{array}{l} \exists b_1, \ldots b_n \in \mathcal{O}_X(X) \text{ s.t.} \\ D(b_i) \text{ are affine open subschemes and } \langle b_1, \ldots, b_n \rangle = \mathcal{O}_X(X) \end{array} \tag{6.24}$$

Pick $x \in X$, and an open affine neighbourhood U of x. Let $Y = X \setminus U$
Take any subscheme structure on Y which corresponds to some quasi-coherent ideal sheaf $\mathcal{I}_Y$
We can consider $\{x\}$ as a closed subscheme: $\operatorname{Spec} \kappa(x)$

(Step 1):
Consider $Y \cup \{x\}$, which is a closed subscheme correspond to an ideal sheaf $\mathcal{I}_{Y \cup \{x\}}$.
We have an exact sequence:

$$0 \to \mathcal{I}_{Y \cup \{x\}} \to \mathcal{I}_Y \to \kappa(x) \to 0 \tag{6.25}$$

(Note that for open $W \subseteq X$, if $x \notin W$ then $\mathcal{I}_{Y \cup \{x\}}(W) = \mathcal{I}_Y(W)$, so $\mathcal{I}_Y \,/\, \mathcal{I}_{Y \cup \{x\}}$ is 0 outside $\{x\}$, i.e. the skyscraper sheaf at x given by $\kappa(x)$, see Definition **??**)

(Step 2):
The long exact sequence induced by s.e.s (**??**) gives

$$0 \to H^0(X, \mathcal{I}_{Y \cup \{x\}}) \to H^0(X, \mathcal{I}_Y) \overset{\alpha}{\twoheadrightarrow} H^0(X, \kappa(x))(= \kappa(x)) \to H^1(X, \mathcal{I}_{Y \cup \{x\}}) = 0 \tag{6.26}$$

$\Rightarrow \quad \exists b \in H^0(X, \mathcal{I}_Y) \subseteq H^0(X, \mathcal{O}_X) = \mathcal{O}_X(X)$ a.t. $\alpha(b) = 1$
$\Rightarrow \quad x \in D(b)$
Also, $b \in H^0(X, \mathcal{I}_Y) \Rightarrow D(b) \cap Y = \emptyset$ (look at this locally)
$\Rightarrow \quad D(b) \subseteq U$
$\Rightarrow \quad D(b)$ is an open affine subscheme

(Step 3):
Now we can cover X with open sets of the form $D(b)$. Since X is Noetherian, X can be covered by finitely many such sets, say $D(b_1), \dots, D(b_n)$
(Proof-unrelated remark: if X is affine, then $\langle b_1, \dots, b_n \rangle = \mathcal{O}_X(X)$)

(Step 4):
We have a surjective morphism

$$\phi : \underbrace{\mathcal{O}_X \oplus \cdots \oplus \mathcal{O}_X}_{n \text{ copies}} \to \mathcal{O}_X \tag{6.27}$$

which defined as follows:
For open $W \subseteq X$

$$\phi_W : \mathcal{O}_X(W) \oplus \cdots \oplus \mathcal{O}_X(W) \quad \to \quad \mathcal{O}_X(W) \tag{6.28}$$
$$\underbrace{(0, \dots, 1, \dots, 0)}_{1 \text{ at } i\text{-th place}} \quad \mapsto \quad b_i \tag{6.29}$$

If W is affine then ϕ_W is surjective.
If $\mathcal{F} = \ker \phi$, then we have an exact sequence

$$0 \to \mathcal{F} \to \mathcal{O}_X \oplus \cdots \oplus \mathcal{O}_X \to \mathcal{O}_X \to 0 \tag{6.30}$$

We have a filtration

$$\mathcal{G}_0 = 0 \subseteq \underbrace{\mathcal{O}_X \oplus 0 \oplus \cdots \oplus 0}_{\mathcal{G}_1} \subseteq \underbrace{\mathcal{O}_X \oplus \mathcal{O}_X \oplus \cdots}_{\mathcal{G}_2} \subseteq \mathcal{G}_n = \underbrace{\mathcal{O}_X \oplus \cdots \oplus \mathcal{O}_X}_{n \text{ copies}} \tag{6.31}$$

We also get a filtration

$$\mathcal{F}_0 = 0 \subseteq \underbrace{\mathcal{F} \cap \mathcal{G}_1}_{\mathcal{F}_1} \subseteq \cdots \subseteq \mathcal{F}_n = \mathcal{F} \cap \mathcal{G}_n = \mathcal{F} \tag{6.32}$$

Now we have an injection

$$\frac{\mathcal{F}_{i+1}}{\mathcal{F}_i} = \frac{\mathcal{F} \cap \mathcal{G}_{i+1}}{\mathcal{F} \cap \mathcal{G}_i} \hookrightarrow \frac{\mathcal{G}_{i+1}}{\mathcal{G}_i} \cong \mathcal{O}_X \tag{6.33}$$

$\Rightarrow$ $\mathcal{F}_{i+1} / \mathcal{F}_i$ is an ideal sheaf on X which is coherent.
$\Rightarrow$ by assumptions $H^p(X, \mathcal{F}_{i+1} / \mathcal{F}_i) = 0$ if $p > 0$

Since $0 \to \mathcal{F}_i \to \mathcal{F}_{i+1} \to \mathcal{F}_{i+1} / \mathcal{F}_i \to 0$ is s.e.s and $\mathcal{F}_0 = 0$, inducitve using the long exact sequence induced by this s.e.s one can see that $H^p(X, \mathcal{F}_i) = 0$ if $p > 0$.
In particular, $H^1(X, \mathcal{F}) = 0$.
$\Rightarrow$ long exact sequence induced by sequence (**??**) that

$$0 \to H^0(X, \mathcal{F}) \to \underbrace{H^0(X, \mathcal{O}_X \oplus \cdots \oplus \mathcal{O}_X)}_{\mathcal{O}_X(X) \oplus \cdots \oplus \mathcal{O}_X(X)} \to \underbrace{H^0(X, \mathcal{O}_X)}_{\mathcal{O}_X(X)} \to H^1(X, \mathcal{F}) \to \cdots \tag{6.34}$$

$\Rightarrow$ b_i generate $H^0(X, \mathcal{O}_X) = \mathcal{O}_X(X)$ □

6.1 Čech cohomology

Definition 6.17
X topological space, $\mathcal{U} = (U_i)_{i \in I}$ an open covering, $\mathcal{F} \in \mathcal{S}h(X)$. Assume that I is well-ordered.
Let $U_{i_0, \ldots, i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$. Now for any $p \geq 0$, define

$$C^p(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0, \ldots, i_p}) \tag{6.35}$$

(if $|I| < p + 1$ then put $C^p(\mathcal{U}, \mathcal{F}) = 0$)
Obviously, any element of $C^p(\mathcal{U}, \mathcal{F})$ can be written as $(s_{i_0, \ldots, i_p})_i$ where $s_{i_0, \ldots, i_p} \in \mathcal{F}(U_{i_0, \ldots, i_p})$

- For any $p > 0$, we can define a homomorphism

$$d^p : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F}) \tag{6.36}$$

$$\text{with} \qquad (d^p s)_{i_0, \ldots, i_{p+1}} = \sum_{l=0}^{p+1} (-1)^l s_{i_0, \ldots, \widehat{i_l}, \ldots, i_{p+1}} |_{U_{i_0, \ldots, i_{p+1}}} \tag{6.37}$$

- It turns out that $d^{p+1} d^p = 0$. Therefore, we get a complex, called <u>Čech complex</u>

$$0 \to C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d^0} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d^1} \cdots \tag{6.38}$$

- Define the Čech cohomology groups

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = \frac{\ker d^p}{\operatorname{Im} d^{p-1}} \tag{6.39}$$

Theorem 6.18
Setting as in **??**. Then, $\check{H}^0(\mathcal{U}, \mathcal{F}) = H^0(X, \mathcal{F}) = \mathcal{F}(X)$

Proof
By definition, $C^0(\mathcal{U}, \mathcal{F}) = \prod_{i \in I} \mathcal{F}(U_i)$ and $[d^0(s)]_{ij} = s_j|_{U_i \cap U_j} - s_i|_{U_i \cap U_j}$ $(i < j)$

$$\Rightarrow \qquad \ker d^0 = \left\{ (s_i) \in C^0(\mathcal{U}, \mathcal{F}) \middle| s_j|_{U_i \cap U_j} = s_i|_{U_i \cap U_j} \text{ if } i < j \right\} \tag{6.40}$$

The sheaf condition for $\mathcal{F}$ $\Rightarrow$ such (s_i) determine uniquely sections of $\mathcal{F}$ on X i.e. $\ker d^0 \cong \mathcal{F}(X)$
We also know that $H^0(X, \mathcal{F}) = \mathcal{F}(X)$ □

Example 6.19
$X = \mathbb{P}^1_k = \operatorname{Proj} k[t_0,t_1]$ where k a field $\quad \mathcal{U} = (U_i)_{i\in\{0,1\}} \quad U_0 = D_+(t_0) \quad , \quad U_1 = D_+(t_1)$
Aim: Calculate $\check{H}^*(\mathcal{U},\mathcal{O}_X)$

By definition,
$$C^0(\mathcal{U},\mathcal{O}_X) = \mathcal{O}_X(U_0)\oplus\mathcal{O}_X(U_1) = k[t_0,t_1]_{(t_0)}\oplus k[t_0,t_1]_{(t_1)} \tag{6.41}$$
$U_0\cap U_1 = D_+(t_0t_1) \quad\Rightarrow\quad C^1(\mathcal{U},\mathcal{O}_X) = \mathcal{O}_X(U_0\cap U_1) = k[t_0,t_1]_{(t_0t_1)}$
$\Rightarrow$ the Čech complex is like
$$0 \xrightarrow{d^{-1}} C^0(\mathcal{U},\mathcal{O}_X) \xrightarrow{d^0} C^1(\mathcal{U},\mathcal{O}_X) \xrightarrow{d^1} 0 \tag{6.42}$$
with
$$C^0(\mathcal{U},\mathcal{O}_X) = k[t_0,t_1]_{(t_0)}\oplus k[t_0,t_1]_{(t_1)} \;\to\; k[t_0,t_1]_{(t_0t_1)} \tag{6.43}$$
$$(p,q) \;\mapsto\; q|_{U_0\cap U_1} - p|_{U_0\cap U_1} \tag{6.44}$$
$$\Rightarrow \qquad \ker d^0 = \left\{(p,q)\middle|\; q|_{U_0\cap U_1} = p|_{U_0\cap U_1}\right\} \tag{6.45}$$
Since X is integral, we may consider all sections inside the function field K of X
Now $p=q$ in K, we must have $p,q\in k$:
We can write $\quad p = \frac{F}{t_0^{d_0}}$ and $q = \frac{G}{t_1^{d_1}}$ where $\deg F = d_0$ and $\deg G = d_1$
We have $\quad \frac{F}{t_0^{d_0}} = \frac{G}{t_1^{d_1}} \quad\Rightarrow\quad Ft_1^{d_1} = Gt_0^{d_0}$ in $k(t_0,t_1)$

$k[t_0,t_1]$ is a UFD $\quad\Rightarrow\quad t_0^{d_0}|F$ and $t_1^{d_1}|G$
$\Rightarrow\quad \exists a_0,a_1,\in k$ s.t. $F = a_0t_0^{d_0}$ and $G = a_1t_1^{d_1}$
$\Rightarrow\quad a_0 = \frac{F}{t_0^{d_0}} = \frac{G}{t_1^{d_1}} = a_1$
$\Rightarrow\quad a_0 = a_1 = p = q \in k$
$\Rightarrow\quad \big(H^0(X,\mathcal{O}_X) =\big)\quad \check{H}^0(\mathcal{U},\mathcal{O}_X) = k$

To compute $\check{H}^1(\mathcal{U},\mathcal{O}_X)$, we need to calculate $\operatorname{Im} d^0$
Now $\operatorname{Im} d^0$ is the sub vector space over k generated by all the elements of $k[t_0,t_1]_{(t_0)}$ and $k[t_0,t_1]_{(t_1)}$
Pick $r\in k[t_0,t_1]_{(t_0t_1)}$ $\quad\exists$ some $E\in k[t_0,t_1]$ s.t. $r = \frac{E}{(t_0t_1)^d}$ for some $d\geq 0$ ($\deg E = 2d$)
It is enough to assume that E is a monomial, say $E = t_0^mt_1^n$
$m+n = 2d \quad\Rightarrow\quad m\geq d$ or $n\geq d$
If $m\geq d$ then $\quad r = \frac{E}{(t_0t_1)^d} = \frac{t_0^{m-d}t_1^n}{t_1^d} \in k[t_0,t_1]_{(t_1)}$
If $n\geq d$, similarly, get $\quad r\in k[t_0,t_1]_{(t_0)}$

$\therefore\quad \operatorname{Im} d^0 = k[t_0,t_1]_{(t_0t_1)} \qquad\Rightarrow\qquad \check{H}^1(\mathcal{U},\mathcal{O}_X) = 0$

Example 6.20
Setting as in Example **??**.
First note that

$$U_0 \cong \operatorname{Spec} k[t_0, t_1]_{(t_0)} = \operatorname{Spec} k[\frac{t_1}{t_0}] \tag{6.46}$$

<u>**Aim**</u>: Calculate $\check{H}^*(\mathcal{U}, \Omega_{X/\operatorname{Spec} k})$
We can see that, using above note (**??**),

$$\begin{cases} \Omega_{X/\operatorname{Spec} k}(U_0) \;=\; \mathcal{O}_X(U_0) \cdot d(\frac{t_1}{t_0}) \;=\; k[t_0, t_1]_{(t_0)} d(\frac{t_1}{t_0}) \\ \Omega_{X/\operatorname{Spec} k}(U_1) \;=\; \mathcal{O}_X(U_1) \cdot d(\frac{t_0}{t_1}) \;=\; k[t_0, t_1]_{(t_1)} d(\frac{t_0}{t_1}) \\ \Omega_{X/\operatorname{Spec} k}(U_0 \cap U_1) = (\Omega_{X/\operatorname{Spec} k}(U_0))_{(t_1/t_0)} \cdot d(\frac{t_1}{t_0}) = k[\frac{t_0}{t_1}, \frac{t_1}{t_0}] d(\frac{t_1}{t_0}) \end{cases} \tag{6.47}$$

The Čech complex is

$$0 \to C^0(\mathcal{U}, \Omega_{X/\operatorname{Spec} k}) \xrightarrow{d^0} C^1(\mathcal{U}, \Omega_{X/\operatorname{Spec} k}) \to 0 \tag{6.48}$$

$$C^0(\mathcal{U}, \Omega_{X/\operatorname{Spec} k}) \;=\; k[t_0, t_1]_{(t_0)}\, d\left(\frac{t_1}{t_0}\right) \oplus k[t_0, t_1]_{(t_1)}\, d\left(\frac{t_0}{t_1}\right) \tag{6.49}$$

$$C^1(\mathcal{U}, \Omega_{X/\operatorname{Spec} k}) \;=\; k[\frac{t_0}{t_1}, \frac{t_1}{t_0}]\, d\left(\frac{t_1}{t_0}\right) \tag{6.50}$$

$$d^0\left(pd\left(\frac{t_1}{t_0}\right), qd\left(\frac{t_0}{t_1}\right)\right) \;=\; qd\left(\frac{t_0}{t_1}\right) - pd\left(\frac{t_1}{t_0}\right) \;=\; \left(\frac{t_0^2}{t_1^2} q + p\right) d\left(\frac{t_1}{t_0}\right) \tag{6.51}$$

where we use the fact

$$\frac{t_0}{t_1}\frac{t_1}{t_0} = 1 \text{ on } U_0 \cap U_1 \quad \Rightarrow \quad \frac{t_0}{t_1} d\left(\frac{t_1}{t_0}\right) + \frac{t_1}{t_0} d\left(\frac{t_0}{t_1}\right) = d(1) = 0 \tag{6.52}$$

So now we have

$$\ker d^0 = \left\{ \left(pd(\frac{t_1}{t_0}), qd(\frac{t_0}{t_1}) \right) \middle|\; \frac{t_0^2}{t_1^2} q + p = 0 \right\} \tag{6.53}$$

Simple calculation as in Example **??** shows that $p = q = 0$

$$\Rightarrow \qquad \ker d^0 = 0 \tag{6.54}$$

$$\Rightarrow \qquad \check{H}^0(\mathcal{U}, \Omega_{X/\operatorname{Spec} k}) = 0 \tag{6.55}$$

To calculate $\check{H}^1(\mathcal{U}, \Omega_{X/\operatorname{Spec} k})$, it is enough to see what $\operatorname{Im} d^0$ is.
It is easy to see that

$$\frac{k[t_0/t_1, t_1/t_0]}{\operatorname{Im} d^0} d(\frac{t_1}{t_0}) \cong kd(\frac{t_1}{t_0}) \tag{6.56}$$

$$\Rightarrow \qquad \check{H}^1(\mathcal{U}, \Omega_{X/\operatorname{Spec} k}) \cong k \tag{6.57}$$

Example 6.21
$X = \mathbb{P}^1_k, \quad U_0 = D_+(t_0), \quad U_1 = D_+(t_1)$ (as in Example **??**)
Let $\mathcal{F}$ be the constant sheaf on X defined by $\mathbb{Z}$:

$$\mathcal{F}(W) = \mathbb{Z} \quad \text{if } W \neq \emptyset \tag{6.58}$$

We get the Čech complex as

$$\begin{tikzcd}
0 \arrow[r] & C^0(\mathcal{U},\mathcal{F}) \arrow[r, "d^0"] \arrow[d, equal] & C^1(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d, equal] & 0 \\
& \mathcal{F}(U_0) \oplus \mathcal{F}(U_1) \arrow[d, equal] & \mathcal{F}(U_0 \cap U_1) \arrow[d, equal] & \\
& \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} & \\
& (m,n) \arrow[r, mapsto] & n-m &
\end{tikzcd}$$

$$\begin{aligned}
\ker d^0 &= \{(m,m) | m \in \mathbb{Z}\} \cong \mathbb{Z} && (6.59)\\
\operatorname{Im} d^0 &= \mathbb{Z} && (6.60)\\
\check{H}^*(\mathcal{U},\mathcal{F}) &= \begin{cases} \mathbb{Z} & * = 0 \\ 0 & * > 0 \end{cases} && (6.61)
\end{aligned}$$

Example 6.22
Let $X = S^1$ (the circle), $U = S^1 \setminus \{b\}, \;\; V = S^1 \setminus \{a\}$
(a, b are the north and south pole)
$U \cap V = S^1 \setminus \{a, b\}$ (topology on X is the one induced by $\mathbb{R}^2$)
Let $\mathcal{F}$ be the constant sheaf on X defined by $\mathbb{Z}$, the Čech complex is

$$\begin{tikzcd}
0 \arrow[r] & C^0(\mathcal{U},\mathcal{F}) \arrow[r, "d^0"] \arrow[d, equal] & C^1(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d, equal] & 0 \\
& \mathcal{F}(U) \oplus \mathcal{F}(V) \arrow[d, equal] & \mathcal{F}(U \cap V) \arrow[d, equal] & \\
& \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} & \\
& (m,n) \arrow[r, mapsto] & (n-m, n-m) &
\end{tikzcd}$$

$$\begin{aligned}
\ker d^0 \cong \mathbb{Z} &\qquad \operatorname{Im} d^0 \cong \mathbb{Z} && (6.62)\\
\check{H}^0(\mathcal{U},\mathcal{F}) = \mathbb{Z} &\qquad \check{H}^1(\mathcal{U},\mathcal{F}) = \mathbb{Z} && (6.63)
\end{aligned}$$

Definition 6.23
X topological space, $\mathcal{U} = (U_i)_{i \in I}$ a finite covering, $\mathcal{F} \in \mathcal{S}h(X)$
For each $i_0 < \cdots < i_p$, let $f_{i_0,\ldots,i_p} : U_{i_0,\ldots,i_p} \hookrightarrow X$ be the inclusion morphism and let

$$\mathcal{F}_{i_0,\ldots,i_p} = (f_{i_0,\ldots,i_p})_* \mathcal{F}|_{U_{i_0,\ldots,i_p}} \tag{6.64}$$

and put

$$\mathcal{C}^p(\mathcal{U},\mathcal{F}) = \prod_{i_0,\ldots,i_p} \mathcal{F}_{i_0,\ldots,i_p} \tag{6.65}$$

(Note that $\mathcal{C}^p(\mathcal{U},\mathcal{F})(X) = C^p(\mathcal{U},\mathcal{F})$)
For similar reason (to Definition **??**), we get a complex of sheaves

$$\mathcal{C}^0(\mathcal{U},\mathcal{F}) \xrightarrow{d^0} \mathcal{C}^1(\mathcal{U},\mathcal{F}) \xrightarrow{d^1} \cdots \tag{6.66}$$

also called the <u>Čech complex</u>

Lemma 6.24
Setting as in Definition **??**. Then, there is a morphism $\mathcal{F} \to \mathcal{C}^0(\mathcal{U}, \mathcal{F})$ s.t. we get an exact sequence

$$0 \to \mathcal{F} \to \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d^0} \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \to \cdots \tag{6.67}$$

Proof
Note that the morphism $\mathcal{F} \to \mathcal{C}^0(\mathcal{U}, \mathcal{F})$ is defined by:
$W \subseteq X$ open and $s \in \mathcal{F}(W)$, then $s \mapsto (s|_{W \cap U_i})_i$

The exactness of $0 \to \mathcal{F} \to \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d^0} \mathcal{C}^1(\mathcal{U}, \mathcal{F})$ is just the sheaf condition for $\mathcal{F}$

To prove that the rest of the sequence is exact we use:

Claim: The induced sequence on stalks is exact

Proof of Claim:
Pick $x \in X$. We can assume that $x \in U_{1,\dots,n}$ where $I = \{1, \dots, n\}$
(because if $x \notin U_j$, we could just ignore the index j)
For each $p \geq 1$, we define a map

$$\begin{aligned} e^p : \mathcal{C}^p(\mathcal{U}, \mathcal{F})_x &\to \mathcal{C}^{p-1}(\mathcal{U}, \mathcal{F})_x & (6.68) \\ (W, s) &\mapsto (W, (t_{i_0,\dots,i_{p-1}})_i) & (6.69) \end{aligned}$$

where $x \in W$ and $s \in \mathcal{C}^p(\mathcal{U}, \mathcal{F})(W) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(W \cap U_{i_0,\dots,i_p}) \quad \Rightarrow \quad s = (s_{i_0,\dots,i_p})_i$
(since $x \in U_{1,\dots,n}$ we could assume $W \subseteq U_{1,\dots,n}$) and

$$t_{i_0,\dots,i_{p-1}} = \begin{cases} s_{1,i_0,\dots,i_{p-1}} & \text{if } i_0 \neq 1 \\ 0 & \text{if } i_0 = 1 \end{cases} \tag{6.70}$$

Note

$$\begin{tikzcd} & \mathcal{C}_x^p \arrow[r, "d^p"] \arrow[dl, "e^p"'] \arrow[d, dashed, "\mathrm{id}_{\mathcal{C}_x^p}"] & \mathcal{C}_x^{p+1} \arrow[dl, "e^{p+1}"] \\ \mathcal{C}_x^{p-1} \arrow[r, "d^{p-1}"'] & \mathcal{C}_x^p & \end{tikzcd}$$

It turns out the one has a formula

$$e^{p+1} d^p + d_0^{p-1} e^p = \mathrm{id}_{\mathcal{C}^p(\mathcal{U}, \mathcal{F})_x} \tag{6.71}$$

Now if $(W, s) \in \mathcal{C}^p(\mathcal{U}, \mathcal{F})_x$ is sent to 0 in $\mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F})_x$ (i.e. $d^p((W, s)) = 0$)
$\Rightarrow$ From formula (**??**), get $d^{p-1} e^p(W, s) = (W, s)$
$\Rightarrow$ $(W, s) \in \mathrm{Im}(\mathcal{C}^{p-1}(\mathcal{U}, \mathcal{F})_x \to \mathcal{C}^p(\mathcal{U}, \mathcal{F})_x)$
$\Rightarrow$ the sequence is exact at $\mathcal{C}^p(\mathcal{U}, \mathcal{F})_x \Rightarrow$ the sequence of stalk is exact ■

□

Theorem 6.25
X topological space, $\mathcal{U} = (U_i)_{i \in I}$ finite covering, $\mathcal{F} \in \mathcal{S}h(X)$ flasque
Then $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ if $p > 0$

Proof
We use exact sequence from Lemma **??**:

$$0 \to \mathcal{F} \to \mathcal{C}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d^0} \mathcal{C}^1(\mathcal{U}, \mathcal{F}) \to \cdots \tag{6.72}$$

$\mathcal{F}$ is flasque $\quad\Rightarrow\quad \mathcal{F}|_{U_{i_0,\ldots,i_p}}$ is flasque
$\Rightarrow\quad \mathcal{F}_{i_0,\ldots,i_p} = (f_{i_0,\ldots,i_p})_* \mathcal{F}|_{U_{i_0,\ldots,i_p}}$ is flasque (c.f. Example sheet 4)
$\Rightarrow\quad \mathcal{C}^p(\mathcal{U},\mathcal{F}) = \bigoplus_{i_0,\ldots,i_p} \mathcal{F}_{i_0,\ldots,i_p}$ is flasque
$\Rightarrow$ sequence (**??**) is a flasque resolution of $\mathcal{F}$

From the usual theory of cohomology we know that if we consider

$$0 \xrightarrow{d^{-1}} \mathcal{C}^0(\mathcal{U},\mathcal{F})(X) \xrightarrow{d^0} \mathcal{C}^1(\mathcal{U},\mathcal{F})(X) \xrightarrow{d^1} \cdots \tag{6.73}$$

$$\Rightarrow \qquad H^p(X,\mathcal{F}) = \frac{\ker d^p}{\operatorname{Im} d^{p-1}} = 0 \quad (p>0) \qquad \text{by Theorem \textbf{??}} \tag{6.74}$$

On the other hand, the sequence (**??**) is just the Čech complex $0 \to \mathcal{F} \to \mathcal{C}^0(\mathcal{U},\mathcal{F}) \xrightarrow{d^0} \mathcal{C}^1(\mathcal{U},\mathcal{F})$ which calculate the Čech cohomology groups

$$\Rightarrow \qquad \check{H}^p(X,\mathcal{F}) = H^p(X,\mathcal{F}) = 0 \qquad (p>0) \tag{6.75}$$

□

Theorem 6.26
X a Noetherian scheme separated over $\operatorname{Spec}\mathbb{Z}$, and $\mathcal{F}$ a quasi-coherent sheaf on X, and $\mathcal{U} = (U_i)_{i\in I}$ a finite cover by open affine schemes. Then

$$H^p(X,\mathcal{F}) \cong \check{H}^p(\mathcal{U},\mathcal{F}) \tag{6.76}$$

Proof
First, we embed $\mathcal{F}$ into a quasi-coherent $\mathcal{G} = \prod_i \mathcal{G}_i$ which is flasque, where we embed $\mathcal{F}|_{U_i}$ into a quasi-coherent flasque sheaf $\mathcal{G}_i'$ and $\mathcal{G}_i$=direct image of $\mathcal{G}_i'$
(Note: $\mathcal{F}|_{U_i} = \widetilde{M}$, then $0 \to M \to J$ with J injection induces $0 \to \widetilde{M} \to \widetilde{J}$ with $\widetilde{J}$ flasque)
It is easy to see that $\mathcal{F} \to \mathcal{G}$ is injection.

Now let $\mathcal{H} = \mathcal{G}\,/\,\mathcal{F} \quad\Rightarrow\quad$ get exact sequence:

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0 \tag{6.77}$$

where $\mathcal{H}$ is quasi-coherent
For any $i_0 < \cdots < i_p$, then

$$0 \to \mathcal{F}(U_{i_0,\ldots,i_p}) \to \mathcal{G}(U_{i_0,\ldots,i_p}) \to \mathcal{H}(U_{i_0,\ldots,i_p}) \to H^1(U_{i_0,\ldots,i_p}, \mathcal{F}|_{U_{i_0,\ldots,i_p}}) = 0 \tag{6.78}$$

$(\mathcal{F}(U_{i_0,\ldots,i_p}) = H^0(U_{i_0,\ldots,i_p}, \mathcal{F}))$
Now $U_{i_0,\ldots,i_p}$ is affine because X is separated (Example sheet 2)
The vanishing follows from the fact that $U_{i_0,\ldots,i_p}$ is affine and $\mathcal{F}|_{U_{i_0,\ldots,i_p}}$ is quasi-coherent
$\Rightarrow$ the direct sum of all sequence like **??** will give an exact sequence of (Čech) complexes:

$$0 \to C^*(\mathcal{U},\mathcal{F}) \to C^*(\mathcal{U},\mathcal{G}) \to C^*(\mathcal{U},\mathcal{H}) \to 0 \tag{6.79}$$

Now Definition 6.2 says that we get a long exact sequence (top row) and get a commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & \check{H}^p(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & \check{H}^p(\mathcal{U},\mathcal{G}) \arrow[r] \arrow[d] & \check{H}^p(\mathcal{U},\mathcal{H}) \arrow[r] \arrow[d] & \check{H}^{p+1}(\mathcal{U},\mathcal{F}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^p(X,\mathcal{F}) \arrow[r] & H^p(X,\mathcal{G}) \arrow[r] & H^p(X,\mathcal{H}) \arrow[r] & H^{p+1}(X,\mathcal{F}) \arrow[r] & \cdots
\end{tikzcd}$$

Now the isomorphism $H^p(X,\mathcal{F}) = \check{H}^p(\mathcal{U},\mathcal{F})$ follows from the facts that the isomorphism holds for $p=0$, and $H^p(X,\mathcal{G}) = 0 = \check{H}^p(\mathcal{U},\mathcal{G})$ because $\mathcal{G}$ is flasque □

Theorem 6.27
k a field and $X = \mathbb{P}^n_k = \operatorname{Proj} S = \operatorname{Proj} k[t_0, \dots, t_n]$, then

(1) $H^0(X, \mathcal{O}_X(d)) =$ vecotr space over k generated by non-negative degree monomials of degree d

(2) $H^n(X, \mathcal{O}_X(d)) \cong H^0(X, \mathcal{O}_X(-d-n-1))$

(3) $H^p(X, \mathcal{O}_X(d)) = 0 \quad$ if $0 < p < n$ or $p > n$

Proof
X is Noetherian and separated over $\operatorname{Spec} k$ ($\Rightarrow$ over $\operatorname{Spec} \mathbb{Z}$)
Put $U_i = D_+(t_i)$ and take $\mathcal{U} = (U_i)_{i=0,\dots,n}$ to be the open affine covering of X

(1) $H^0(X, \mathcal{O}_X(d)) = \mathcal{O}_X(d)(X)$
Sheaf condition $\Rightarrow$ Every element of $\mathcal{O}_X(d)(X)$ is uniquely determined by a system (s_i) where $s_i \in \mathcal{O}_X(d)(U_i)$ with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$
$\mathcal{O}_X(d)(U_i) = S(d)_{(t_i)}$
$\Rightarrow \quad \forall i,\ s_i = \dfrac{f_i}{t_i^{l_i}}$ s.t. f_i homogeneous and $\deg f_i = l_i + d$ (as an element in S)

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \quad \Leftrightarrow \quad \frac{f_i}{t_i^{l_i}} = \frac{f_j}{t_j^{l_j}} \text{ inside the field } k(t_0, \dots, t_n)$$
$$\Leftrightarrow \quad f_i t_j^{l_j} = f_j t_i^{l_i} \text{ in } S$$

$\Rightarrow \quad t_i^{l_i} | f_i$ and $t_j^{l_j} | f_j$
$\Rightarrow \quad \exists g \in S$ s.t. $g = \frac{f_i}{t_i^{l_i}}$ $\forall i$ with $\deg g = d$
$\Rightarrow \quad \mathcal{O}_X(d)(X)$ is generated as a vector space over k by the homog. polynomials of degree d

(2) The Čech complex is as

$$\cdots \to C^{n-1}(\mathcal{U}, \mathcal{O}_X(d)) \xrightarrow{d^{n-1}} C^n(\mathcal{U}, \mathcal{O}_X(d)) \to 0 \tag{6.80}$$

with
$$C^n(\mathcal{U}, \mathcal{O}_X(d)) = \mathcal{O}_X(d)(U_{0,\dots,n}) = S(d)_{(t_0 \cdots t_n)} \tag{6.81}$$

$\forall \alpha \in C^n(\mathcal{U}, \mathcal{O}_X(d))$, have $\alpha = \dfrac{f}{(t_0 \cdots t_n)^l}$ s.t. $\deg f = (n+1)l + d$ in S

Assume that f is a monomial, say $f = t_0^{m_0} \cdots t_n^{m_n}$
If $m_i \geq l$ for some $i \quad \Rightarrow \quad \alpha \in \operatorname{Im} d^{n-1}$
$\Rightarrow \quad$ assume that $m_i < l \ \forall i$
Can also assume $m_i = 0$ for some i, otherwise we can make l smaller
$\Rightarrow \quad (n+1)l + d = \deg f \leq n(l-1)$
$\Rightarrow \quad d \leq -l - n \leq -1 - n$

Now suppose that $d \geq -n-1$
$\Rightarrow \quad \alpha \in \operatorname{Im} d^{n-1}$ unless $l = 1, d = -n-1$
$\Rightarrow \quad \deg f = 0$ and $C^n(\mathcal{U}, \mathcal{O}_X(d))/\operatorname{Im} d^{n-1} \cong H^n(X, \mathcal{O}_X(d)) = \left\langle \dfrac{1}{t_0 \cdots t_n} \right\rangle$
$\Rightarrow \quad H^n(X, \mathcal{O}_X(-n-1)) \cong k$ and $H^n(X, \mathcal{O}_X(d)) = 0$ if $d > -n-1$
We accept the claim of (2) for $d < -n-1$

(3) We may assume that $n \geq 2$. Let Y be the closed subscheme defined by $\langle t_n \rangle$
If $\mathcal{I}_Y$ is the ideal sheaf of Y, then we have an s.e.s:

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to \pi_* \mathcal{O}_Y \to 0 \tag{6.82}$$

where $\pi : Y \to X$ closed immersion
The sequence comes from the exact sequence:

$$0 \to S(-1) \to S \to S/\langle t_n \rangle \cong k[t_0, \dots, t_{n-1}] \to 0 \tag{6.83}$$

(Note $S(-1) \cong \langle t_n \rangle$ via $f \mapsto f t_n$)
In particular,

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{S(-1)} \arrow[r] \arrow[d, "\text{isom}"] & \widetilde{S} \arrow[r] \arrow[d, "\text{isom}"] & \widetilde{S/\langle t_n \rangle} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_X(-1) \cong \mathcal{I}_y \arrow[r] & \mathcal{O}_X \arrow[r] & \pi_* \mathcal{O}_Y(d) \arrow[r] & 0
\end{tikzcd}$$

If we tensor the sequence with $\mathcal{O}_X(d)$, we get exact sequence

$$0 \to \mathcal{O}_X(d-1) \to \mathcal{O}_X(d) \to \pi_* \mathcal{O}_Y(d) \to 0 \tag{6.84}$$

$\Rightarrow$ get an exact sequence

$$\begin{aligned} &H^{n-1}(X, \mathcal{O}_X(d)) \to \underbrace{H^{n-1}(X, \pi_* \mathcal{O}_Y(d))}_{V_4} \\ &\xrightarrow{\alpha^{n-1}} \underbrace{H^n(X, \mathcal{O}_X(d-1))}_{V_5} \to \underbrace{H^n(X, \mathcal{O}_X(d))}_{V_6} \to H^n(X, \pi_* \mathcal{O}_Y(d)) = 0 \to 0 \end{aligned} \tag{6.85}$$

We also have an exact sequence

$$0 \to \underbrace{H^0(X, \pi_* \mathcal{O}_Y(d-1))}_{V_1} \to \underbrace{H^0(X, \mathcal{O}_X(d))}_{V_2} \to \underbrace{H^0(X, \pi_* \mathcal{O}_Y(d))}_{V_3} \xrightarrow{\alpha^0} H^1(X, \mathcal{O}_X(d-1)) \to \cdots \tag{6.86}$$

From calculation of the dimension of $V_1, \dots, V_6$, see that α^{n-1} is injective and $\alpha^0 = 0$
Knowing that $H^p(X, \pi_* \mathcal{O}_Y(d)) = 0$ if $0 < p < n-1$ by induction it follows that the maps $\beta^p : H^P(X, \mathcal{O}_X(d-1)) \to H^p(X, \mathcal{O}_X(d))$ are isomorphisms
The map comes from the map

$$\begin{tikzcd}
\mathcal{O}_X(d-1)(U_{i_0, \dots, i_p}) \arrow[r] \arrow[d, "\text{isom}"] & \mathcal{O}_X(d)(U_{i_0, \dots, i_p}) \arrow[d, "\text{isom}"] \\
S(d-1)_{(t_{i_0}, \dots, t_{i_p})} \arrow[r, "\times t_n"] & S(d)_{(t_{i_0}, \dots, t_{i_p})}
\end{tikzcd}$$

(remember $S(-1) \to S$ is given by multiplication by t_n)
$\Rightarrow$ β^p is given by multiplication by t_n
We will prove that if $w \in H^p(X, \mathcal{O}_X(d))$ then $t_n^l w = 0$ for some l
This then implies β^p must be just zero, i.e. $H^p(X, \mathcal{O}_X(d)) = 0$ if $0 < p < n$

Claim: If $w \in H^p(X, \mathcal{O}_X(d))$ then $t_n^l w = 0$ for some l
Proof of Claim:
Put $\mathcal{F} = \bigoplus_{d \in \mathbb{Z}} \mathcal{O}_X(d) \cong \widetilde{\bigoplus_{d \in \mathbb{Z}} S(d)}$
Now it turns out that

$$\mathcal{F}(U_{i_0, \dots, i_p}) = \left(\bigoplus S(d) \right)_{(t_{i_0}, \dots, t_{i_p})} \cong S_{t_{i_0}, \dots, t_{i_p}} \tag{6.87}$$

$\Rightarrow$ Čech complex looks like

$$\begin{array}{ccccccc} 0 & \to & C^0(\mathcal{U},\mathcal{F}) & \to & C^1(\mathcal{U},\mathcal{F}) & \to & \cdots \\ & & \| & & \| & & \\ & & \prod_{i_0} S_{t_{i_0}} & & \prod_{i_0<i_1} S_{t_{i_0}t_{i_1}} & & \end{array} \tag{6.88}$$

If we localise the complex (**??**) at t_n, then we get

$$0 \to \prod S_{(t_{i_0}t_n)} \to \prod S_{(t_{i_0}t_{i_1}t_n)} \to \cdots \tag{6.89}$$

This is the Čech complex of $\mathcal{F}|_{U_n}$ with covering $U_0 \cap U_n, \dots U_{n-1} \cap U_n, U_n$
$\Rightarrow$ $H^p(U_n, \mathcal{F}|_{U_n}) = 0$ if $p > 0$ because U_n is affine. Moreover,

$$H^p(X,\mathcal{F})_{t_n} = H^p(U_n, \mathcal{F}|_{U_n}) = 0 \tag{6.90}$$

So any $w \in H^p(X,\mathcal{F})$ satisfy $t_n^l w = 0$ for some $l \gg 0$ ■

□

(End of Examinable material)

From now on, k is a fixed algebraically closed field

Definition 6.28 (1) X is an integral open subscheme of projecitve scheme over $\operatorname{Spec} k$. Assume that $\mathcal{O}_X$ is a regular ring of dimension d for any closed point $x \in X$

- A Weil divisor on X is an object $M = \sum m_i M_i$ where $m_i \in \mathbb{Z}$ and $M_i = \overline{\mu_i}$ with $\mu_i \in X$ s.t. $\mathcal{O}_{\mu_i}$ is of dimension one (this is a DVR)
- Let K be the function field of X and $f \in K^*$. We define the Weil divisor of f as

$$(f) = \sum_{\substack{\mu \in X \\ \mathcal{O}_\mu \text{ is of dim 1}}} v_\mu(f)\overline{\mu} \tag{6.91}$$

  (note that since $\mathcal{O}_\mathcal{M}$ is DVR, then we have a corresponding valuation $v_\mathcal{M} : K \setminus \{0\} \to \mathbb{Z}$)
- Note that (f) is a finite sum. Since X is Noetherian it is enough to check the finiteness on open affine subschemes. Let $U = \operatorname{Spec} R \subseteq X$ be an open affine subscheme. K is just the function field of R hence we can write $f = \frac{a}{b}$ with $a, b \in R$
  Now if $v_\mu(f) \neq 0$ for some $\mu \in U$ then $\mu \in V(\langle a \rangle) \cup V(\langle b \rangle) = V(\langle ab \rangle)$
  There are only finitely many such μ in $V(\langle ab \rangle)$ hence in U

(2) X is as in (1) but X is projective and $d = 1$
If $M = \sum m_i M_i$ is a Weil divisor on X, we define its degree as $\deg M = \sum m_i$
Fact: here $\deg(f) = 0$

Theorem 6.29
Setting as in Definition **??**(2). Then there is a one-to-one correspondence between Cartier and Weil divisor

Proof
($\leftarrow$:) Suppose that D is a Cartier divisor given by a system (U_i, f_i) on U_i we have the Weil divisor (of U_i), (f_i). Putting all the (f_i) give a Weil divisor on X (need to take closure of (f_i)).

From the definition of a Cartier divisor, $\frac{f_i}{f_j}$ and $\frac{f_j}{f_i} \in \mathcal{O}_X^*(U_i \cap U_j)$
This exactly mean that (f_i) and (f_j) coincide on $U_i \cap U_j$

($\rightarrow$:) Conversely, if $M = \sum m_\alpha M_\alpha$ is a Weil divisor, then we define a Cartier divisor as follows:
Pick a closed point $x \in X$. The multiplicity of x in M is just m_α if $x = M_\alpha$ (if x does not appear in M then $m_\alpha = 0$)
Since $\mathcal{O}_x$ is just a DVR then the maximal ideal $\mathfrak{m}_x \subseteq \mathcal{O}_x$ is generated by one element t.
Now put $f = t^{m_\alpha} \quad \Rightarrow \quad (f) = M$ in an open affine neighbourhood of x
X Noetherian $\quad \Rightarrow \quad$ it can be covered by finitely many such open affine scheme
$\Rightarrow \quad$ we get a system (U_i, f_i) s.t. U_i cover X and $(f_i) = M$ on U_i
$(f_i) = (f_j)$ on $U_i \cap U_j \quad \Rightarrow \quad \frac{f_i}{f_j}$ and $\frac{f_j}{f_i}$ are in $\mathcal{O}_X^*(U_i \cap U_j)$
$\Rightarrow \quad (U_i, f_i)$ defines a Cartier divisor $\square$

Definition 6.30
X as in Definition **??**(2). If D is a Cartier divisor on X. Define $\deg D$ to be the $\deg M$ when M is the corresponding Weil divisor

- If $D \sim D'$ (i.e. $D = (U_i, f_i), \exists D' = (U_i, f f_i)$), then $\deg D = \deg D'$

Theorem 6.31 (Duality Theorem)
X as in Definition **??**(1) and X projective. Then $\exists$ a Cartier divisor K_X s.t. if D is any Cartier divisor then

$$\dim_k H^p(X, \mathcal{O}_X(D)) = \dim_k H^{d-p}(X, \mathcal{O}_X(K_X - D)) \tag{6.92}$$

(Moreover, the vector space have finite dimesion)

(Proof omitted)

Theorem 6.32 (Riemann-Roch Theorem)
(c.f. Theorem 0.15)
X as in Definition **??**(2). Then for any Cartier divisor D on X we have

$$\dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^0(X, \mathcal{O}_X(K_X - D)) = \deg D + 1 - \dim_k H^0(X, \mathcal{O}_X(K_X)) \tag{6.93}$$

Proof
Let D be a Cartier divisor. Let $x \in X$ be a closed point.
Let $M = \sum m_i M_i$ be the Weil divisor corresponding to D
Let D_x be the Cartier divisor associated to x considered as a Weil divisor.

Claim: There is an exact sequence

$$0 \to \mathcal{O}_X(D) \to \mathcal{O}_X(D + D_x) \to k(x) \to 0 \tag{6.94}$$

where $k(x)$ is the skyscraper sheaf at x given by k

Proof of Claim:

If D is given by a system (U_i, f_i) where U_i are open affine, and if $W \subset X$ is any open subset, then

$$\begin{array}{rcll} \mathcal{O}_X(D)(W) & \to & \mathcal{O}_X(D + D_x)(W) & (6.95) \\ \| & & \| & (6.96) \\ \{h \in K \big| h f_i \in \mathcal{O}_X(U_i \cap W)\ \forall i\} & \to & \{e \in K \big| e f_i g_i \in \mathcal{O}_X(U_i \cap W)\ \forall i\} & \\ h & \mapsto & h & (6.97) \end{array}$$

is a well-defined homomorphism, because:
the Weil divisor of D_x is just x, then $g_i \in \mathcal{O}_X(U_i)$ and so if $h f_i \in \mathcal{O}_X(W \cap U_i)$ then $h f_i g_i \in \mathcal{O}_X(W \cap U_i)$

This type of maps define an injection morphism $\mathcal{O}_X(D) \hookrightarrow \mathcal{O}_X(D + D_x)$
$\Rightarrow \quad \dfrac{\mathcal{O}_X(D + D_x)}{\mathcal{O}_X(D)}$ is the skyscraper sheaf at x defined by k because:
if $W \subseteq X$ open affine and $x \notin W$ then $g_i \in \mathcal{O}_X^*(W \cap U_i)$

This short exact sequence induces long exact sequence:

$$\begin{aligned} 0 \to H^0(X, \mathcal{O}_X(D)) \to H^0(X, \mathcal{O}_X(D + D_x)) \to H^0(X, k(x)) \\ \to H^1(X, \mathcal{O}_X(D)) \to H^1(X, \mathcal{O}_X(D + D_x)) \to H^1(X, k(x)) = 0 \to \cdots \end{aligned}$$

Also, by Duality Theorem, $H^1(X, \mathcal{O}_X(D)) \cong H^0(X, \mathcal{O}_X(K_X - D))$
and $H^1(X, \mathcal{O}_X(D + D_x)) \cong H^0(X, \mathcal{O}_X(K_X - D - D_x))$

By linear algebra,

$$\begin{aligned} \dim_k H^0(X, \mathcal{O}_X(D + D_x)) - \dim_k H^0(X, \mathcal{O}_X(K_X - D - D_x)) \\ = \dim_k H^0(X, \mathcal{O}_X(D)) - \dim_k H^0(X, \mathcal{O}_X(K_X - D)) + \dim_k H^0(X, k(x)) - \dim_k H^1(X, k(x)) \end{aligned} \tag{6.98}$$

this calculation implies that the theorem holds for $D \quad \Leftrightarrow \quad$ the theorem holds for $D + D_x$ since $\deg(D + D_x) = \deg D + 1$
So by removing or adding points to M, we can redeem the theorem to the case $M = 0$

But in this case, $\dim_k H^0(X, \mathcal{O}_X(D)) = 1$ (exercise) and $\deg D = 0$ $\square$

Remark. When $k = \mathbb{C}$, $H^0(X, \mathcal{O}_X(K_X))$ is the genus of curve represented by scheme (variety) X
Hint for exercise: Prove $H^0(X, \mathcal{O}_X(D)) \cong H^0(X, \mathcal{O}_X)$ and that $\mathcal{O}_X(X)$ is a f.g. k-algebra which is a field